Qualifying Exam
Logic
August 29, 1996

Instructions: If you signed up for Recursion Theory, do two E and two R problems. If you signed up for Model Theory, do two E and two M problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Suppose that A_n is a subset of the ordinal $(\omega_1)^3$ for each $n \in \omega$, and suppose that $\bigcup_{n\in\omega} A_n = (\omega_1)^3$. Prove that some A_n has order type $(\omega_1)^3$. Here, $(\omega_1)^3$ denotes ordinal exponentiation (ω_1 to the third power).

E2. Let $\mathcal{L} = \{<\} \cup \{c_q : q \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rationals. Let $\mathfrak{A}$ be the natural model for $\mathcal{L}$; that is, $A = \mathbb{Q}$, each c_q is interpreted as q, and $<$ is interpreted as the usual order on $\mathbb{Q}$. Prove that the theory of $\mathfrak{A}$ is decidable.

E3. Describe a finite set of axioms, S, in a finite language, such that for each finite $n > 0$: S has a model of size n iff n is a power of 2.

In the Recursion Theory problems, φ_e is the e^{th} partial recursive function of one variable, using some standard enumeration.

R1. Prove that there are sets $A_n \subseteq \omega$, for $n \in \omega$, such that each A_n is recursive in A_m, but no A_n is primitive recursive in any A_m unless $n = m$.

R2. Let S be the set of all $e \in \omega$ such that $dom(\varphi_e)$ is an initial segment of ω (possibly, all of ω). Prove that S is not recursive in $0'$.

R3. Prove that there is a total recursive function f such that each $\varphi_{f(x)}$ is total and $\varphi_{f(x)}(x) = f(x) + x$.

M1. Let $\mathfrak{A}$ be a structure for $\mathcal{L}$, and let U be a unary predicate symbol. Assume $|U_{\mathfrak{A}}| = \mathbf{c}$ (where $\mathbf{c} = 2^{\aleph_0}$). Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$, such that $|U_{\mathfrak{B}}| = \mathbf{c}^+$. Note that we are assuming nothing about $|\mathfrak{A}|$ or the size of $\mathcal{L}$.

M2. Let $\mathcal{L} = \{<\}$. Describe a complete theory T in $\mathcal{L}$ such that

1. In every model $\mathfrak{A}$ for $\mathcal{L}$, $<_A$ totally orders A
2. There are $2^{\aleph_0}$ different 1-types consistent with T.

M3. Let T be a complete theory with infinite models. Assume that T has some model with an automorphism σ of order 2 (that is, σ^2 is the identity but σ isn't). Let $\mathfrak{A}$ be any model of T. Prove that $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that $\mathfrak{B}$ has an automorphism of order 2.

Answers to Logic Qual January 1996

E1. Identify $(\omega_1)^3$ with $\omega_1 \times \omega_1 \times \omega_1$, ordered lexically.

E2. Prove that it has a decidable complete set of axioms: dense total order without endpoint, plus the ordering of the c_q.

E3. S can say that $<$ totally orders the universe, I is a proper initial segment of $<$, and P defines a bijection from the universe onto the power set of I. So, I is a 1-place predicate and P is a 2-place predicate; the bijection is really $x \mapsto \{y \in I : P(x, y)\}$. You can say the map is onto using finitely many axioms by saying that for each set in the range, the lexically next set is in the range as well.

R1. Just let all the A_n be recursive, and construct them in ω steps to defeat all possible primitive recursive computations of one from another.

R2. S is a complete Π^0_2 set.

R3. f can be a constant function.

M1. Form an elementary chain of $\mathbf{c}^+$ models, starting with $\mathfrak{A}$. At limits, take unions. At successors, take an ultrapower using an ultrafilter on ω.

M2. List the rationals, $\mathbb{Q}$, as $\{q_n : n \in \omega\}$. Form A by replacing each q_n by a sequence of n points, and let T be the theory or $\mathbb{Q}$.

M3. Let $\mathfrak{C}$ be the model with the automorphsim. Then, embed $\mathfrak{A}$ into a $|\mathfrak{A}|$ - saturated elementary extension of $(\mathfrak{C}, \sigma)$.