Qualifying Exam
Logic
January 16, 1997

Instructions: If you signed up for Recursion Theory, do two E and two R problems. If you signed up for Model Theory, do two E and two M problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathcal{L} = \{f, E\}$, where f is a unary function symbol and E is a binary relation symbol. Let T be the theory in $\mathcal{L}$ whose axioms say that E is an equivalence relation with exactly three equivalence classes, $\forall x E(x, f(x))$, f is a one-one onto function, and f has no finite cycles (that is, for $n = 1, 2, \ldots$, T has the axiom $\forall x(x \neq f^n(x))$. Prove that T is complete but not finitely axiomatizable.

E2. If $f : \omega \times \omega \to \{0, 1\}$, define $r = r_f : \omega \to \mathbb{R}$ by:

$$r(e) = \sum_{n<\omega} f(e, n) \cdot 2^{-n} \quad .$$

Prove that there is a computable f such that $\{e : r_f(e) \in \mathbb{Q}\}$ is undecidable. $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers.

E3. *Without* using the Axiom of Choice, prove that there is a bijection from the set of real numbers onto the set of irrational numbers.

In the Recursion Theory problems, φ_e is the e^{th} partial recursive function of one variable, using some standard enumeration.

R1. Prove that there are sets $A_n \subseteq \omega$, for $n \in \omega$, such that each A_n is recursive in A_m, but no A_n is primitive recursive in any A_m unless $n = m$.

R2. Let S be the set of all $e \in \omega$ such that $dom(\varphi_e)$ is an initial segment of ω (possibly, all of ω). Prove that S is not recursive in $0'$.

R3. Prove that there is a total recursive function f such that each $\varphi_{f(x)}$ is total and $\varphi_{f(x)}(x) = f(x) + x$.

M1. Without assuming the Continuum Hypothesis, do the following:

1. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, $|A| = |B| = \aleph_2$, and such that there are no ultrafilters $\mathcal{U}, \mathcal{V}$ on ω with $\mathfrak{A}^\omega/\mathcal{U}$ isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$.

2. Describe two structures, $\mathfrak{A}$ and $\mathfrak{B}$, for a finite language, such that: $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, $|A| = |B| = \aleph_2$, and such that $\mathfrak{A}^\omega/\mathcal{U}$ is isomorphic to $\mathfrak{B}^\omega/\mathcal{V}$ whenever $\mathcal{U}, \mathcal{V}$ are any non-principal ultrafilters on ω.

M2. Let $\mathfrak{M}$ be an infinite saturated $\mathcal{L}$-structure. Assume $X \subseteq M$ is definable with parameters $\vec{a} \in M^{<\omega}$; that is, for some $\mathcal{L}$-formula $\theta(x, \vec{y})$:

$$X = \{m \in M : \mathfrak{M} \models \theta(m, \vec{a})\} \ .$$

Assume also that every automorphism f of $\mathfrak{M}$ satisfies $f(X) = X$. Prove that X is definable without parameters; that is, for some $\mathcal{L}$-formula $\psi(x)$:

$$X = \{m \in M : \mathfrak{M} \models \psi(m)\} \ .$$

M3. Let $\mathcal{L}$ contain the symbol $<$, and let $\mathfrak{A}$ be an $\mathcal{L}$-structure in which $<_{\mathfrak{A}}$ is a total order with no largest element. Prove that $\mathfrak{A}$ has an elementary extension, $\mathfrak{B}$ such that:

1. $\mathfrak{B}$ has a non-trivial automorphism.

2. $<_{\mathfrak{B}}$ has uncountable cofinality (that is, every countable subset of B is bounded).

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E1. *Completeness*: Let $\mathfrak{A}, \mathfrak{B}$ be any two models of T. By the Löwenheim-Skolem Theorem, there are countable models $\mathfrak{A}', \mathfrak{B}'$ elementarily equivalent to $\mathfrak{A}, \mathfrak{B}$. By the Compactness Theorem, there are elementary extensions $\mathfrak{A}'', \mathfrak{B}''$ which each have $\aleph_0$ orbits in each equivalence class. Therefore, $\mathfrak{A}''$ is isomorphic to $\mathfrak{B}''$. Thus, $\mathfrak{A}, \mathfrak{B}$ are elementarily equivalent, so T is complete. *Not finitely axiomatizable*: Use the fact that every finite subset of T has a finite model.

E2. Fix any irrational $x \in (0, 1)$. Let $A \subseteq \omega \times \omega$ be any decidable set such that $\{e : \exists n[(e, n) \in A]\}$ is undecidable. Let $f(e, n)$ be the e^{th} bit in the binary expansion of x if $(e, n) \in A$ and $f(e, n) = 0$ otherwise.

E3. Apply the Schröder-Bernstein Theorem. Prove that $|\mathbb{R}\backslash\mathbb{Q}| \leq |\mathbb{R}| \leq |\mathcal{P}(Q)| = |2^\omega| \leq |\mathbb{R}\backslash\mathbb{Q}|$. Here, $|X| \leq |Y|$ means that there is an injection from X into Y. For the last $\leq$, just use the standard construction of a perfect set of irrational numbers.

R1. Just let all the A_n be recursive, and construct them in ω steps to defeat all possible primitive recursive computations of one from another.

R2. S is a complete Π^0_2 set.

R3. f can be a constant function.

M1. For 1: Let $\mathfrak{A}$ and $\mathfrak{B}$ both code three total orders. In $\mathfrak{A}$, the orders have cofinalities $\omega, \omega_1, \omega_2$. In $\mathfrak{B}$, they all have cofinality ω.

For 2: Let $\mathfrak{A}$ and $\mathfrak{B}$ consist of just a set (unary relation). In $\mathfrak{A}$, the set has size $\aleph_0$. In $\mathfrak{B}$, the set has size $\aleph_1$.

M2. Let Γ be the complete $\mathcal{L}$-type of $\vec{a}$ in $\mathfrak{M}$. Consider the set of formulas in $\mathcal{L}_{\vec{a}}$:

$$\Sigma(\vec{y}) = \Gamma(\vec{y}) \cup \Gamma(\vec{a}) \cup \{\exists x[\theta(x, \vec{y}) \leftrightarrow \neg\theta(x, \vec{a})]\}.$$

If $\Sigma(\vec{y})$ is consistent, then by saturation, it is realized in $\mathfrak{M}$ by some $\vec{d}$, and by saturation again, there is an automorphism f which satisfies $f(\vec{a}) = \vec{d}$. By the definition of Σ, $f(X) \neq X$. Therefore, Σ is inconsistent. Thus, there is a $\varphi(\vec{y}) \in \Gamma(\vec{y})$ such that

$$\mathfrak{M} \models \forall x\vec{y}\vec{z}[[\varphi(\vec{y}) \wedge \varphi(\vec{z})] \rightarrow [\theta(x, \vec{y}) \leftrightarrow \theta(x, \vec{z})]] \quad .$$

so

$$X = \{m \in M : \mathfrak{M} \models \exists\vec{y}[\varphi(\vec{y}) \wedge \theta(m, \vec{y})]\} \quad .$$

M3. By the usual Eherenfeuct-Mostowski argument, get an elementary extension with a non-trivial automorphism. Then, add a name for the automorphism to the language and take elementary extensions ω_1 times to construct $\mathfrak{B}$.