Qualifying Exam
Logic
August 27, 1998

Instructions: If you signed up for Recursion Theory, do two E and two R problems. If you signed up for Set Theory, do two E and two S problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let T be a finitely axiomatizable theory in a finite language $\mathcal{L}$. Assume that for each sentence θ of $\mathcal{L}$, either $T \cup \{\theta\}$ has a finite model or $T \cup \{\theta\}$ is inconsistent. Prove that the set of sentences of $\mathcal{L}$ which are provable from T is decidable.

E2. $[S]^\omega$ denotes the family of all countably infinite subsets of S. Prove that if $|S| = \omega_2$, then there is an $\mathcal{A} \subseteq [S]^\omega$ such that $|\mathcal{A}| = \omega_2$ and $\mathcal{A}$ satisfies: $\forall x \in [S]^\omega \exists a \in \mathcal{A}[x \subseteq a]$.

Don't assume CH. *Remark (and hint).* This is also true for ω_1, but it's easier. If $\mathcal{A} = \{\alpha : \omega \leq \alpha < \omega_1\}$, then $|\mathcal{A}| = \omega_1$ and $\forall x \in [\omega_1]^\omega \exists a \in \mathcal{A}[x \subseteq a]$.

E3. Prove that there is an additive subgroup of the reals which is totally imperfect, ie. it and its complement intersect every uncountable closed set of reals.

You may use without proof that there are exactly continuum many closed sets of reals and any uncountable closed set of reals has cardinality the continuum.

R1. The definitions below describe recursive procedures which define one partial recursive function in terms of another. For each definition, apply the Recursion Theorem to obtain a fixed point. What is the fixed point?

$$\text{(a)}\quad \varphi_{h(e)}(n) = \begin{cases} 0 & \text{if } n = 0, \\ \varphi_e(n-1) & \text{if } n > 0 \text{ and } \varphi_e(n-1)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

$$\text{(b)}\quad \varphi_{k(e)}(n) = \begin{cases} n & \text{if } \varphi_e(n) \downarrow \neq n, \\ \uparrow & \text{otherwise.} \end{cases}$$

φ_e is the e^{th} partial recursive function of one variable, using some standard enumeration. A fixed point of a total recursive function f is a function φ_e such that $\varphi_e = \varphi_{f(e)}$

R2. Show that there is a 3-r.e. set which is not of d.r.e. Turing degree.

A set X is d.r.e. iff there are recursively enumerable sets Y and Z such that $X = Y - Z$, the set-theoretic difference of Y and Z. A degree is d.r.e. if it contains a d.r.e. set. A set X is 3-r.e. iff there are recursively enumerable sets W, Y and Z such that $X = W - (Y - Z)$.

R3. Given a 1-generic set A, let A_0 and A_1 be the sets of even and odd members of A, respectively. Show that the Turing degrees of A_0 and A_1 form a minimal pair.

A set A is 1-generic iff for any r.e. set S of strings there exists $\sigma \subset A$ such that either $\sigma \in S$ or $\forall \tau \supseteq \sigma\ (\tau \notin S)$. Strings are finite sequences of 0's and 1's, and $\sigma \subset A$ means that σ is an initial segment of the characteristic function of A. A minimal pair of degrees is a pair of nonzero degrees $\mathbf{a}$ and $\mathbf{b}$ such that the only degree $\mathbf{x} \leq \mathbf{a}, \mathbf{b}$ is the recursive degree $\mathbf{0}$.

S1. Let M be a countable transitive model for ZFC. Prove that if g is Cohen-generic over M and $f \in 2^\omega \cap M$, then $f + g$ is Cohen-generic over M.

Here, $(f+g)(n) = f(n)+g(n) \pmod 2$. A function $g : \omega \to 2$ is *Cohen-generic* over M iff the filter $G = \{p \in Fn(\omega, 2) : p \subset g\}$ is $Fn(\omega, 2)$-generic over M.

S2. Let $(T, <)$ be a tree of height ω_1, with root node 1. Assume that for all $x \in T$ and all $\alpha < \omega_1$, there is is a $y \in T$ above x at level $\geq \alpha$. Consider $(T, <)$ as a partial order for forcing by reversing $<$; so that 1 is now the largest element. **Also** assume that forcing with T doesn't collapse ω_1. Prove that forcing with T doesn't add any ω-sequences of ordinals.

S3. Assume MA and $2^{\aleph_0} \geq \aleph_3$. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\omega)$ with $|\mathcal{A}| = |\mathcal{B}| = \aleph_2$. Assume that whenever $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$, with $|\mathcal{A}'| = |\mathcal{B}'| = \aleph_1$, there is an $x \subseteq \omega$ such that $\forall a \in \mathcal{A}'[a \subseteq^* x]$ and $\forall b \in \mathcal{B}'[b \perp x]$. Prove that there is an $x \subseteq \omega$ such that $\forall a \in \mathcal{A}[a \subseteq^* x]$ and $\forall b \in \mathcal{B}[b \perp x]$.

Here, $a \subseteq^* x$ means that $a \backslash x$ is finite, and $a \perp x$ means that $a \cap x$ is finite.

Answers

E1. Note that we can effectively enumerate all theorems of T and we can also effectively enumerate all sentence of $\mathcal{L}$ which are true in some finite model of T. To decide whether T proves θ we simultaneously enumerate these two lists and wait until θ appears on the first list or $\neg\theta$ appears on the second.

E2. Without loss of generality assume that $S = \omega_2$. For each $\alpha < \omega_2$ let $\mathcal{A}_\alpha$ be an increasing ω_1 sequence of countable sets whose union covers α. Then $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha < \omega_2\}$ works.

E3. Prove the following

Lemma: Suppose $G, X, C \subseteq R$, G is subgroup of R, G and X are disjoint, $|G \cup X| < c$, and $|C| = c$. Then there exists $g \in C$ such that the subgroup generated by $G \cup \{g\}$ is disjoint from X.

Let C_α for $\alpha < c$ be all uncountable closed sets of reals. Use the Lemma to construct G_α, X_α so that G_α is a subgroup of R, G_α and X_α are disjoint, $\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$ and $X_\alpha \subseteq X_\beta$, $|G_\alpha \cup X_\alpha| = |\alpha| + \omega$, and $G_{\alpha+1}$ and $X_{\alpha+1}$ both meet C_α. $G = \bigcup\{G_\alpha : \alpha < c\}$ is the group desired.

R1. The procedures are recursive since the conditions for defining the functions $\varphi_{h(e)}(n)$ and $\varphi_{k(e)}(n)$ (when these are defined) are Σ_1. (a) the constant function 0 (b) the empty function.

R2. The following strategies can be put together in a finite injury argument: We build a 3-r.e. set A and ensure $A \neq \Phi(W)$ or $W \neq \Psi(A)$ for all p.r. functionals Φ and Ψ and all d.r.e. sets W.

Pick a fresh witness x and keep it out of A.

Wait for $\Phi(W; x) = A(x)$ and $W \restriction use(x) = \Psi(A) \restriction use(x)$. (*)

Restrain A up to use of Ψ and put x into A.

Wait for (*) to happen again.

Restrain A also up to the new use of Ψ and take x out of A, forcing W back.

Now put x back into A, and W can't change back again.

R3. Suppose $B = \{e_0\}^{A_0} = \{e_1\}^{A_1}$ and define

$$S = \{\sigma : \exists m \;\; \{e_0\}^{\sigma_0}(m) \neq \{e_1\}^{\sigma_1}(m)\}$$

It must be that there exists $\sigma \subset A$ such that $\forall \tau \supseteq \sigma$ $(\tau \notin S)$. Then $B(m) = i$ iff there exists $\tau \supseteq \sigma$ such that $\{e_0\}^{\tau_0}(m) = i$. Thus B is recursive.

S1. Suppose $D \in M$ is dense in $Fn(\omega, 2)$ and H is the filter generated by $f + g$. Let

$$D^* = \{t \in Fn(\omega, 2) : s + t \in D \text{ where } s = f|_{\mathrm{dom}(t)}\}$$

Then D^* is dense and in M. G meets D^* implies that H meets D.

S2. Note that the generic filter G is just a chain in the tree, of order type ω_1. Suppose that p forces that τ is an ω-sequence of ordinals. In $V[G]$: for each n, choose a $q_n \in G$ extending p which decides $\tau(n)$; then choose an $r \in G$ which extends all the q_n. Then, back in V: use r to decide all of τ.

S3. Let $\mathbb{P}$ be the partial order of disjoint pairs (A, B) such that A is a finite union of elements of $\mathcal{A}$ mod finite and B is a finite union of elements of $\mathcal{B}$ mod finite. Stronger conditions increase in both coordinates. To see that $\mathbb{P}$ is ccc, given (A_α, B_α) for $\alpha < \omega_1$ apply the above condition to find X such that each $A_\alpha \subseteq^* X$ and $B_\alpha \perp X$. Then for any α and β where the pairs of finite sets $A_\alpha \setminus X = A_\beta \setminus X$ and $X \cap B_\alpha = X \cap B_\beta$ agree, the corresponding conditions are compatible.