Qualifying Exam
Logic (Recursion Theory)
January 15, 1999

Instructions: Do three E and three R problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. For distinct $x, y \in 2^\omega$, define $l(x, y)$ to be the least n such that $x(n) \neq y(n)$, and give 2^ω the metric d where $d(x, y) = 2^{-l(x,y)}$. Prove that for any two countable dense sets $X, Y \subseteq 2^\omega$ there exists a bijection $f : X \to Y$ such that $l(f(u), f(v)) = l(u, v)$ for any two distinct $u, v \in X$.

E2. Let M be a model which is countably universal, that is, M has a countable vocabulary and every countable model which is elementarily equivalent to M is elementarily embeddable in M. Let $\psi(x, y)$ be a formula of L such that for every $a \in M$ the set $S_a = \{b \in M : M \models \psi(a, b)\}$ is finite. Prove that there is a finite n such that $|S_a| \leq n$ for every $a \in M$.

E3. Prove in ZF that there is a singular cardinal κ such that for each regular $\lambda < \kappa$ there are arbitrarily large cardinals $\mu < \kappa$ of cofinality λ.

R1. Let E be the set of $e \in \omega$ such that φ^2_e is the characteristic function of a relation which totally orders ω isomorphically to the rationals. Prove that E is a complete Π^0_2 set – that is, E is Π^0_2 and every Π^0_2 set is many-one reducible to E.

R2. Let R be a decidable well-order of ω and let α be the order type of R. Prove that there is another decidable well-order, S, of the same order type α, such that the (unique) isomorphism between R, S is not computable.

R3. One version of the Recursion Theorem states that for every total recursive h, there is an e such that $\varphi_{h(e)} = \varphi_e$. Note that e is never unique, but the function φ_e may be; also, φ_e may or may not be a total function. Let T_h be the set of all total functions f such that for some e, $f = \varphi_{h(e)} = \varphi_e$. For example, if $h(x) = x$ for all x, then $|T_h| = \aleph_0$. Prove that for each finite n, there is a total recursive h such that $|T_h| = n$.

S1. Let M be a countable transitive model for ZFC. Inside M, form P, the set of infinite subsets of ω, and order it by the subset relation, $\subseteq$.

Now, let G be P-generic over M, and prove that in $M[G]$: G is a non-principal ultrafilter on ω and has the Ramsey property.

Here, an ultrafilter U on ω has the *Ramsey property* iff given any partition $f : [\omega]^2 \to 2$, there is a set $H \in U$ such that f is constant on all pairs from H.

S2. A *super-Aronszajn* tree is a tree $(T, <)$ of height ω_2 such that every level is *countable* and there are no chains in T of size $\aleph_2$. Prove that there are no super-Aronszajn trees.

S3. Assume that $2^{\aleph_0} \geq \aleph_3$. Prove that there are $a, b \subseteq \omega$ such that $a \notin L[b]$ and $b \notin L[a]$. Here, $L[x]$, for $x \subseteq \omega$, is the class of sets constructible from x; we form it as we do L, but starting with $L_0[x] = \{x\} \cup \omega$ instead of $L_0 = \emptyset$.

Answers

E1.

This a back and forth argument similar to that one used by Cantor to show that any two countable dense linear orders without end points are order isomorphic. The following lemma is all that it is needed to do each step.

Lemma. Suppose $U, V \subseteq 2^\omega$ are finite, $h : U \to V$ is a level preserving bijection, and $u \in 2^\omega \setminus U$. Then there exists a nonempty clopen set $C \subseteq 2^\omega$ such that for any $v \in C$ the map extending h which takes u to v is level preserving.

proof. Let $x \in U$ maximize $l(x, u)$ for all $x \in U$. Suppose $l(x, u) = n$. There cannot be any $y \in U$ with $l(x, y) = n$, since then $l(y, u) > n$. Let

$$C = \{v \in 2^\omega : \ \ l(h(x), v) = n\}$$

Since h is level preserving, C is disjoint from V. Suppose $v \in C$. Obviously, $l(x, u) = l(h(x), v) = n$. If $y \in U$ is different from x, then it is easy to check that $l(y, u) = l(y, x)$ and $l(h(y), v) = l(h(y), h(x))$. Hence $l(y, u) = l(y, x) = l(h(x), h(y)) = l(h(y), v)$. qed.