Qualifying Exam
Logic (Set Theory)
August 31, 1999

Instructions:

Do all six problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let D_n, for $n \in \omega$, be subsets of the plane ($\mathbb{R} \times \mathbb{R}$). Assume that $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$, and that each D_n is dense (i.e., meets every nonempty open set). Prove that there is a dense $E \subseteq \mathbb{R} \times \mathbb{R}$ such that each $E \backslash D_n$ is finite.

E2. Let Σ be the following set of axioms in $\mathcal{L} = \{<, P\}$, where $<$ is a binary predicate symbol and P is a unary predicate symbol:

1. $<$ is a dense total order without first or last element.
2. $\forall xy\, [[x < y \wedge P(y)] \to P(x)]$.

Prove that

(a) Σ has only finitely many complete extensions.
(b) $\{\varphi : \Sigma \vdash \varphi\}$ is decidable.

E3. Show there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}\{r\}$ is a Bernstein set for every $r \in \mathbb{R}$. Here, a Bernstein set is an $X \subset \mathbb{R}$ such that neither X nor $\mathbb{R} \backslash X$ contains an uncountable closed (in $\mathbb{R}$) subset.

S1. Assume MA. Fix $\kappa < 2^{\aleph_0}$, and suppose that $A_\alpha \subset \omega$, for $\alpha < \kappa$, and each A_α has asymptotic density 0. Prove that there is a $B \subset \omega$ such that B has asymptotic density 0 and $A_\alpha \subseteq^* B$ for each α. Here, $A \subseteq^* B$ means that $A \backslash B$ is finite, and B has asymptotic density 0 iff $\lim_{n\to\infty} |B \cap n|/n = 0$.

S2. Call $E \subseteq \mathbb{R}$ *distance-unique* iff for each $x, y, u, v \in E$: If $y - x = v - u \neq 0$ then $y = v$ and $x = u$. Let P be the statement that $\mathbb{R}$ is the union of countably many distance-unique sets.

a. Prove that there is an uncountable distance-unique set.
b. Prove that CH implies P.
c. Prove that P implies CH.

Hint for c: Fix $A \subset \mathbb{R}$ and $t_\alpha \in \mathbb{R}$ for $\alpha < \omega_2$ such that $|A| = \aleph_1$ and such that the $A + t_\alpha$ are all disjoint. Show that if E is distance-unique, then $|\{\alpha : |E \cap (A + t_\alpha)| \geq 2\}| \leq \aleph_1$.

S3. Let $\kappa > \omega$ be regular. Assume that $\{\alpha < \kappa : L(\alpha) \models Z\}$ is stationary in κ. Prove that κ is inaccessible in L. Here, Z = Zermelo set theory ($ZF-$ Replacement).

Qualifying Exam
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(a) Σ has only finitely many complete extensions.

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R1. Prove that there is a decidable total order $\lhd$ of ω such that the theory of $(\omega; \lhd)$ is not decidable. The *theory* of $(\omega; \lhd)$ is the set of sentences in $\lhd, =$ which are true in $(\omega; \lhd)$.

R2. Let $W_e = \mathrm{dom}(\varphi_e)$. Prove that there is an infinite decidable $S \subseteq \omega$ such that W_e is an infinite subset of S for all $e \in S$, and such that $W_a \subseteq W_e$ whenever $a \in W_e$ and $e \in S$.

R3. A set $A \subseteq \omega$ is *simple* iff A is r.e., $\omega \backslash A$ is infinite, and A meets every infinite r.e. set. For $V \subseteq \omega \times \omega$, let $V_e = \{x : (e, x) \in V\}$. Prove or disprove: There exists an r.e. set V such that $\{V_e : e \in \omega\}$ is precisely the set of all *non*-simple r.e. sets.

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THE ANSWERS

E1. Let $\{U_n : n \in \omega\}$ list all open balls with rational center and radius. Choose $p_n \in U_n \cap D_n$, and let $E = \{p_n : n \in \omega\}$.

E2. Consider the following five sentences:

$\psi_0 :\ \forall x\,[P(x)]$
$\psi_1 :\ \forall x\,[\neg P(x)]$
$\psi_2 :\ \exists y \forall x\,[P(x) \Leftrightarrow x < y]$
$\psi_3 :\ \exists y \forall x\,[P(x) \Leftrightarrow x \leq y]$
$\psi_4 :\ \neg\psi_0 \wedge \neg\psi_1 \wedge \neg\psi_2 \wedge \neg\psi_3$

Each $\Sigma \cup \{\psi_i\}$ is complete (it's $\aleph_0$-categorical), and $\Sigma \vdash \varphi$ iff $\Sigma \cup \{\psi_i\} \vdash \varphi$ for each $i < 5$.

E3. Let $\mathbb{R} = \{y_\eta : \eta < \mathfrak{c}\}$. Let $\mathfrak{c} \times \mathfrak{c} = \{(\xi_\beta, \eta_\beta) : \beta < \mathfrak{c}\}$. List the uncountable closed sets as $\{P_\xi : \xi < \mathfrak{c}\}$. Choose $x_\beta \in P_{\xi_\beta} \setminus \{x_\alpha : \alpha < \beta\}$, and let $f(x_\beta) = y_{\eta_\beta}$. Then for each $y = y_\eta$ and each $P = P_\xi$, there is a β such that $\xi_\beta = \xi$ and $\eta_\beta = \eta$, and then, for $x = x_\beta$: $f(x) = y$ and $x \in P_\xi$. Thus, $f^{-1}\{y\} \cap P \neq \emptyset$ for every y, P.

R1. For each n, let φ_n be the sentence which says that there are elements $a_1 \lhd a_2 \lhd \cdots \lhd a_n$ such that there are no elements between each a_i, a_{i+1} but such that a_1 is a limit from the left and a_n is a limit from the right. One can arrange for $\{n : (\omega; \lhd) \models \varphi_n\}$ to be a non-decidable r.e. set.

R2. Using the Recursion Theorem, find a primitive recursive function f such that $x < y \Rightarrow f(x) < f(y)$, and such that $W_{f(x)} = \{f(y) : y > x\}$. Let $S = \mathrm{ran}(f)$. Then for each $e = f(x) \in S$, we have $W_e \subset S$. If $a = f(y) \in W_e$, then $y > x$, so $W_a = \{f(z) : z > y\} \subset W_e$. S is primitive recursive because f is increasing.

To get f: The Recursion Theorem says that given a partial recursive g, there is a d such that for all x: $\varphi_d(x) = g(d, x)$. If g happens to be primitive recursive, then φ_d will be primitive recursive also.

Here, we want $f = \varphi_d$ to have the property that for each n: $n \in \mathrm{dom}(\varphi_{\varphi_d(x)})$ iff $\exists y > x\,[n = \varphi_d(y)]$. We also want φ_d to be increasing. So, let $g(x, d)$ be a Gödel number of the partial recursive function:

$$n \mapsto \mu\langle y, C\rangle\,[T(d, y, C)\ \&\ U(C) = n\ \&\ y > x]\ \ .$$

Choose g by primitive recursion on x so that $g(x + 1, d) > g(x, d)$.

R3. To construct such a V: Let $V_{2^a3^b}$ consist of those $\varphi_a(n)$ such that: the values $\varphi_a(0), \ldots, \varphi_a(n)$ and $\varphi_b(0), \ldots, \varphi_b(n)$ are all defined, the $\varphi_b(0), \ldots, \varphi_b(n)$ are all distinct, and $\forall i, j \le n\, [\varphi_a(i) \ne \varphi_b(j)]$. Then $V_{2^a3^b}$ will either be finite or disjoint from the infinite r.e. set $\{\varphi_b(j) : j \in \omega\}$. Let the V_e, for e not of the form 2^a3^b, list all the cofinite sets.

S1. B will be $\bigcup_{\alpha<\kappa}(A_\alpha \backslash F(\alpha))$, where $F : \kappa \to \omega$. To force B to have asymptotic density 0, elements of $\mathbb{P}$ will be pairs, $p = \langle f_p, h_p \rangle$, where $f_p \in Fn(\kappa, \omega)$, $h_p \in Fn(\omega, \omega)$, and

$$\forall i \in \mathrm{dom}(h_p)\, \forall n \ge h_p(i) \qquad \left| \bigcup \{(A_\alpha \backslash f_p(\alpha)) \cap n : \alpha \in \mathrm{dom}(f_p)\} \right| < n \cdot 2^{-i} \quad .$$

Meeting κ dense sets yields an F (and hence B), plus an $H : \omega \to \omega$, such that $\forall n \ge H(i)\, [|B \cap n|/n \le 2^{-i}]$.

S2. For (a,b): Let $G_\alpha \nearrow \mathbb{R}$ (for $\alpha < \mathfrak{c}$), where $G_0 = \emptyset$, $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$ for limit γ, and each G_α is a divisible subgroup of $\mathbb{R}$ (whenever $\alpha > 0$). If $|E \cap (G_{\alpha+1} \backslash G_\alpha)| \le 1$ for all α, then E is distance-unique. Under CH, one can have each G_α countable, let $G_{\alpha+1} \backslash G_\alpha = \{a^n_\alpha : n < \omega\}$, and $E_n = \{a^n_\alpha : \alpha < \omega_1\}$.

For (c), to verify the hint, suppose that $|\{\alpha : |E \cap (A + t_\alpha)| \ge 2\}| = \aleph_2$. Then there are distinct $a, b \in A$ and $\alpha \ne \beta$ such that E contains the four elements: $x := a + t_\alpha$, $y := b + t_\alpha$, $u := a + t_\beta$, $v := b + t_\beta$. Then $y - x = b - a = v - u$.

S3. $C := \{\alpha : L(\alpha) \prec L(\kappa)\}$ is club in κ. If $(\kappa = \lambda^+)^L$, then the Power Set Axiom is false in $L(\kappa)$, so $\forall \alpha \in C\, [L(\alpha) \not\models Z]$.