Qualifying Exam
Logic (Model Theory)
September 1, 2000

Instructions:
Do all six of the problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Consider the following two axiom of choice type principles:

LO. Every set can be linearly ordered.

M. For any set X the partial order $(P(X), \subseteq)$ (the power set of X ordered by inclusion) contains a maximal linearly ordered (under inclusion) subset.

Prove in set theory without the axiom of choice (ZF) that LO and M are equivalent.

E2. Write a finite set Σ of axioms in a finite language such that for all $n \in \omega$: Σ has a model of size n iff $n = k!$ for some k.

E3. Call a real number r *computable* iff the sequence of digits in the decimal representation of r is computable. Prove that there is a computable function $f : \omega \to \omega \backslash \{0\}$ such that $\sum_n \frac{1}{f(n)}$ is finite and not computable.

M1. Suppose that $\mathcal{L}$ is a language which contains among its symbols a unary relation symbol $\underline{\omega}$ and constants $\ulcorner 0 \urcorner, \ulcorner 1 \urcorner, \ulcorner 2 \urcorner, \ldots, \ulcorner n \urcorner, \ldots \quad (n \in \omega)$. An $\omega-$*model* for $\mathcal{L}$ is a model in which $\underline{\omega}$ is interpreted by ω and each $\ulcorner n \urcorner$ by n.

Find $\aleph_2$ first-order sentences in some language $\mathcal{L}$ such that every $\aleph_1$ of them has an ω-model, but the whole collection doesn't.

Hint: Consider linear orders with countable initial segments.

M2. Let $\mathcal{B} = (B; \wedge, \vee, {}', U)$, where $(B; \wedge, \vee, {}')$ is an atomless boolean algebra and U is an ultrafilter on $\mathcal{B}$ (viewed as a unary predicate). Prove that the theory of $\mathcal{B}$ is decidable.

M3. Let $\mathcal{L}$ be a first-order language and T an $\mathcal{L}$-theory. Assume that T is model complete and universally axiomatizable. Fix a model $\mathcal{A} = (A; \ldots)$ of T and a function $f : A \to A$ which is definable in $\mathcal{A}$ without using parameters. Show that f is piecewise given by $\mathcal{L}$-terms; that is, there are finitely many $\mathcal{L}$-terms $t_1(x), \ldots, t_k(x)$ each with at most one free variable x such that

$f(a) \in \{t_1(a), t_2(a), \ldots, t_k(a)\}$ for every $a \in A$.

Qualifying Exam
Logic (Set Theory)
September 1, 2000

Instructions:

Do all six of the problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Consider the following two axiom of choice type principles:

LO. Every set can be linearly ordered.

M. For any set X the partial order $(P(X), \subseteq)$ (the power set of X ordered by inclusion) contains a maximal linearly ordered (under inclusion) subset.

Prove in set theory without the axiom of choice (ZF) that LO and M are equivalent.

E2. Write a finite set Σ of axioms in a finite language such that for all $n \in \omega$: Σ has a model of size n iff $n = k!$ for some k.

E3. Call a real number r *computable* iff the sequence of digits in the decimal representation of r is computable. Prove that there is a computable function $f : \omega \to \omega \setminus \{0\}$ such that $\sum_n \frac{1}{f(n)}$ is finite and not computable.

S1. Suppose that $S \subseteq \omega_1$ is stationary. Prove that there exist pairwise disjoint stationary S_n such that

$$S = \bigcup_{n<\omega} S_n$$

S2. Let κ be an uncountable regular cardinal. Let V_κ be the sets of rank less than κ and H_κ the sets whose transitive closure has cardinality less than κ.

Prove that $V_\kappa = H_\kappa$ iff κ is a strongly inaccessible cardinal.

S3. Assume MA_{ω_1}. Prove that for any partial order $\mathbb{P}$ with the ccc and cardinality ω_1 and for any family $\langle D_\alpha : \alpha < \omega_1 \rangle$ of dense subsets of $\mathbb{P}$, there exists $\langle G_n : n < \omega \rangle$ such that each G_n is a $\mathbb{P}$-filter and $\mathbb{P} = \bigcup_{n<\omega} G_n$ and for every $\alpha < \omega_1$ and for every $n < \omega$ we have $G_n \cap D_\alpha \neq \emptyset$.

Hint: Consider the direct product of countably many copies of $\mathbb{P}$.

Qualifying Exam
Logic (Computability Theory)
September 1, 2000

Instructions:
Do all six of the problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Consider the following two axiom of choice type principles:

LO. Every set can be linearly ordered.

M. For any set X the partial order $(P(X), \subseteq)$ (the power set of X ordered by inclusion) contains a maximal linearly ordered (under inclusion) subset.

Prove in set theory without the axiom of choice (ZF) that LO and M are equivalent.

E2. Write a finite set Σ of axioms in a finite language such that for all $n \in \omega$: Σ has a model of size n iff $n = k!$ for some k.

E3. Call a real number r *computable* iff the sequence of digits in the decimal representation of r is computable. Prove that there is a computable function $f : \omega \to \omega \backslash \{0\}$ such that $\sum_n \frac{1}{f(n)}$ is finite and not computable.

C1. Construct a computable binary relation on ω which is a tree (graph without infinite loops) with an infinite branch A such that A is not computable but A meets every infinite computably enumerable set.

C2. A set A is called **retraceable** if there is a partial computable function f such that for all $x \in A$, if x is the least element of A, then $f(x) = x$ and if $x \in A$ is not the least element of A, $f(x)$ is equal to the largest element of A which is strictly less than x.

a) Show there is a noncomputable retraceable set.

Hint: Consider paths through the full binary branching tree $\{0,1\}^{<\omega}$.

b) Show that every noncomputable retraceable set is immune. Recall that a set A is immune if A is infinite but does not contain an infinite computably enumerable set.

C3. Given a nonzero Turing degree b, build a degree a such that a and b are Turing incomparable but $a' = b'$.

Answers

E1. To see that LO implies M, let $(X, \leq)$ be any linear ordering. Define $I \subseteq X$ is an initial segment of X iff $x \leq y \in I$ implies $x \in I$ for any $x, y \in X$. Show that the set of initial segments of X is a maximal linearly ordered subset of $P(X)$. To see that M implies L, let C be a maximal linearly ordered subset of $P(X)$ and show that $x \leq y$ defined by $x \in Y$ implies $y \in Y$ for every $Y \in C$ is a linear ordering of X.

E2. Let Σ axiomatize the set $n = \{0, 1, \ldots, n-1\}$, together with the *graphs* of $+, \cdot$ and the successor and factorial functions. To say $n = k! = k \cdot (k-1)!$, you can say that there is a bijection from the universe (n) onto $k \times (k-1)!$.

E3. Define f so that for each n, the n^{th} digit of $\sum_n \frac{1}{f(n)}$ is 2 if the n^{th} computation halts and 1 if the n^{th} computation doesn't halt.

M1. Let $\mathcal{L}$ have a symbol for $<$ and a constant $\ulcorner\alpha\urcorner$ for each $\alpha \leq \omega_2$, plus a binary function F, plus a constant c. Consider the structure

$$(\omega_2 + 1; <, F, \alpha)_{\alpha \leq \omega_2}.$$

In this structure, F is chosen so that when $\omega \leq \alpha < \omega_1$, the map $x \mapsto F(\alpha, x)$ defines a 1-1 map from α into ω and when $\omega_1 \leq \alpha < \omega_2$, the map $x \mapsto F(\alpha, x)$ defines a 1-1 map from α into ω_1. Let Σ be the complete diagram of this structure, plus the sentence $c < \ulcorner\omega_2\urcorner$, plus the sentence $c \neq \ulcorner\alpha\urcorner$ for each $\alpha \leq \omega_2$.

Then, in an ω-model for Σ, every $\alpha < \omega_1$ must be standard. But then ω_1 must be standard as well, since the theory says that every element below ω_1 can be injected into ω. Repeating this argument with the ordinals between ω_1 and ω_2, we see that ω_2 must be standard as well. Thus, Σ cannot have an ω-model, but there will be an ω-model for Σ with any of the sentences $c \neq \ulcorner\alpha\urcorner$ deleted.

M2. The theory of atomless boolean algebras with an ultrafilter is complete; in fact, $\aleph_0$-categorical.

M3. By the compactness theorem, it is enough to show that for any $a \in A$ there is an $\mathcal{L}$-term $t(x)$ with at most one free variable x such that $f(a) = t(a)$. Fix $a \in A$, and let

$$B := \{t(a) : \ t(x) \text{ is an } \mathcal{L}\text{-term with at most one free variable } x\}.$$

Then $B \subseteq A$, and if F is an n-ary function symbol of $\mathcal{L}$ and $c \in B^n$, then $f(c) \in B$. Hence B is the underlying set of the substructure $\mathcal{B}$ of $\mathcal{A}$ obtained by interpreting each symbol S of $\mathcal{L}$ as $S^{\mathcal{B}} := S^{\mathcal{A}}|_B$. Since T is universally axiomatizable we get $\mathcal{B} \models T$, and since T is model complete it follows that $\mathcal{B}$ is an elementary substructure of $\mathcal{A}$. Thus, $f(a)$ being the unique image of a under f in $\mathcal{A}$ implies that $f(a) \in B$, that is, $f(a) = t(a)$ for some $\mathcal{L}$-term $t(x)$ with at most one free variable x, as desired.

S1. First show that every stationary set $S \subseteq \omega_1$ can be splitt into two stationary subsets. Let $x_\alpha \in 2^\omega$ be distinct for $\alpha < \omega_1$. For each $n < \omega$ and $i = 0, 1$ let

$$S_i^n = \{\alpha \in S : x_\alpha(n) = i\}$$

Then for some n both S_0^n and S_1^n are stationary, otherwise, if not, for each n there exists i_n and closed and unbounded C_n such that $C_n \cap S_{i_n} = \emptyset$. But since $C = \bigcap_{n<\omega} C_n$ is closed and unbounded it meets S in at least two points α, β. By definition this would imply that for every n $x_\alpha(n) = i_n = x_\beta(n)$ which would make them equal, contradiction. Since every stationary set is splittable into two stationary sets it is now easy by induction to split any stationary set into infinitely many stationary subsets, i.e., split S into $S_0 \cup S^1$, split S^1 into $S_1 \cup S^2$, split S^2 into $S_2 \cup S^3$,and continue.

S2. Suppose that κ is strongly inaccessible. To see that $V_\kappa \subseteq H_\kappa$, prove by induction that for every $\alpha < \kappa$ that $V_\alpha \in H_\kappa$. Suppose $\alpha < \kappa$ and by induction $V_\alpha \in H_\kappa$. Therefor, $|V_\alpha| = \beta < \kappa$, and $|V_{\alpha+1}| = 2^\beta < \kappa$ and since it is transitive we have $V_{\alpha+1} \in H_\kappa$. For a limit ordinal $\lambda < \kappa$, if we have that $V_\alpha \in H_\kappa$ for each $\alpha < \lambda$, then $|V_\alpha| < \kappa$ and hence V_λ has cardinality less than κ since κ is regular. Since its also transitive we have that $V_\lambda \in V_\kappa$. Next show that for any regular infinite κ that $H_\kappa \subseteq V_\kappa$. If not, there exists X with minimal rank such that $X \in H_\kappa$ but $X \notin V_\kappa$. By the minimality of its rank we may assume for all $y \in X$ that there exists $\alpha < \kappa$ with $y \in V_\alpha$. Since κ is regular and X has cardinality less than κ it follows there exists $\beta < \kappa$ such that $X \subseteq V_\beta$ and hence $X \in V_\kappa$, a contradiction. For the other direction, if κ is not strongly inaccessible, then for some $\alpha < \kappa$ we have that $2^\alpha \geq \kappa$. But since $\alpha \subseteq V_\alpha$ we have that $|V_{\alpha+1}| \geq 2^\alpha \geq \kappa$ and hence V_κ is not a subset of H_κ.

S3. Let $\sum_{n<\omega} \mathbb{P}$ be the direct product of countably many copies of $\mathbb{P}$. MA_{ω_1} implies that $\sum_{n<\omega} \mathbb{P}$ has the ccc. For each $\alpha < \omega_1$ and $n < \omega$ define

$$E_\alpha^n = \{p \in \sum_{n<\omega} \mathbb{P} : p_n \in D_\alpha\}$$

Also for each $p \in \mathbb{P}$ define

$$F_p = \{p \in \sum_{n<\omega} \mathbb{P} : \exists n < \omega \;\; p_n = p\}$$

It is easy to check that each E^n_α and each F_p is dense. So by MA_{ω_1} there exists G a $\sum_{n<\omega}\mathbb{P}$ filter which meets all E^n_α for $\alpha < \omega_1$ and $n < \omega$ and all F_p for $p \in \mathbb{P}$. For each $n < \omega$ define $G_n = \{p_n : p \in G\}$. Each G_n is a $\mathbb{P}$ filter meeting all D_α and since G meets each F_p their union covers $\mathbb{P}$.

C1. Assume at stage n in the construction we have a finite connected tree T_n with designated root 0, and a designated maximal path p_n starting at 0. Find the least $e < n$ such that $W_{e,n}$ does not meet p_n, there exists a maximal path p in T_n starting at 0 which meets it, and for any $e' < e$ if $W_{e',n}$ meets p_n then it also meets $p_n \cap p$. Take such a p which agrees with p_n as long as possible if there is one, otherwise let $p = p_n$. Construct T_{n+1} by attaching two new nodes to the end of p. Let p_{n+1} be p plus one of these two nodes. The p_n converge to an infinite path which meets every infinite c.e. set and which is coinfinite and hence not computable.

C2. For part (a), fix a computable coding of $\{0,1\}^{<\omega}$ and consider the set of paths P through $\{0,1\}^{<\omega}$. Each $X \in P$ is a retraceable set with the retracing function given by $f(\emptyset) = \emptyset$, $f(\alpha * 0) = \alpha$, and $f(\alpha * 1) = \alpha$ for all $\alpha \in \{0,1\}^{<\omega}$. The solution follows since $|P| = 2^{\aleph_0}$, but there are only countably many computable sets.

For part (b), assume that A is a retraceable set with retracing function f. Let $W \subset A$ be an infinite c.e. set. We show that A is computable. To decide if $x \in A$, enumerate W until a number $y > x$ enters W. The iterates of f on y (i.e. $f(y), f(f(y)), \ldots$) list all the elements of A less than y in decreasing order.

C3. Fix a nonzero degree b and a set $B \in b$. We build a set A by specifying finite initial segments f_s of its characteristic function, and we insure that $\deg(A)$ has the required properties. The requirements are as follows.

$$\begin{array}{ll} R_e: & A \neq \varphi_e^B \\ S_e: & B \neq \varphi_e^A \\ T_e: & (\exists \sigma \subset A)\Big(\varphi_e^\sigma(e)\downarrow \vee (\forall \tau \supset \sigma)\, \varphi_e^\tau(e)\uparrow\Big) \end{array}$$

The R_e and S_e requirements make A and B Turing incomparable. After the construction, we will verify that T_e together with a coding of B' into A makes $A' \equiv_T B'$.

Construction:

Stage 0: Set $f_0 = \emptyset$.

Stage s+1=4e+1 (Satisfy R_e): Let $n = |f_s|$. Use B' as an oracle to determine if $\varphi_e^B(n)\downarrow = 0$. If so, set $f_{s+1}(n) = 1$. If not, set $f_{s+1}(n) = 0$.

Stage s+1=4e+2 (Satisfy S_e): Use $0'$ as an oracle to test if

$$\exists\sigma,\tau \supset f_s \exists x, t\Big(\varphi^{\sigma}_{e,t}(x)\downarrow \neq \varphi^{\tau}_{e,t}(x)\Big).$$

If not, let $f_{s+1} = f_s$. Otherwise, look at a fixed computable list of all tuples $\langle\sigma,\tau,x,t\rangle$. Pick the least tuple off this list which satisfies the condition above. Use B as an oracle to check if $B(x) = \varphi^{\sigma}_{e}(x)$. If so, let $f_{s+1} = \tau$ and if not, let $f_{s+1} = \sigma$.

Stage s+1=4e+3 (Satisfy T_e): Use $0'$ as an oracle to test if

$$\exists\sigma \supset f_s \exists t(\varphi^{\sigma}_{e,t}(e)\downarrow).$$

If so, take the least σ for which this holds (as above, this means the least relative to some fixed computable list of all pairs $\langle\sigma,t\rangle$), and let $f_{s+1} = \sigma$. If not, let $f_{s+1} = f_s$.

Stage s+1=4e+4 (Code B' into A): Let $n = |f_s|$. Use B' as an oracle and set $f_{s+1}(n) = B'(e)$.

End of construction

Let A be such that $\chi_A = \cup f_s$. (Notice that the stages $4e+4$ guarantee that $\cup f_s$ is total.) At stage $4e+1$, we clearly satisfy R_e. To see that S_e is satisfied, suppose that $B = \varphi^{A}_{e}$. At stage $4e+2$, there must not have been appropriate strings σ and τ, for if there were such strings, we would have diagonalized. It follows that B is computable (giving a contraction) since $B(x) = \varphi^{\sigma}_{e}(x)$ for every string $\sigma \supset f_s$ for which $\varphi^{\sigma}_{e}(x)\downarrow$.

We verify that $A' \equiv_T B'$. First, since the construction can all be done using B' as an oracle, the sequence f_s, $s \in \omega$, is B' computable. Since $e \in A'$ if and only if the answer to our initial question at stage $4e+3$ is yes, B' can determine if $e \in A'$. Therefore, $A' \leq_T B'$.

To see that $B' \leq_T A'$, we need to show that the sequence f_s, $s \in \omega$, is A' computable. This is sufficient since $e \in B'$ if and only if $f_{s+1}(n) = 1$, where $s = 4e+4$ and $n = |f_s|$. To see that the sequence f_s is A' computable (in fact, it is $A \oplus 0'$ computable), we proceed by induction, checking the cases according to the construction.

If $s+1 = 4e+1$ or $s+1 = 4e+4$, then $|f_{s+1}| = |f_s| + 1$ and if $n = |f_s|$, then $f_{s+1}(n) = A(n)$. Therefore, with oracle A we can determine f_{s+1} from f_s. If $s+1 = 4e+3$, then we used only a $0'$ ($\leq_T A'$) oracle to determine f_{s+1} from f_s. Finally, if $s+1 = 4e+2$, then $0'$ ($\leq_T A'$) can determine if there are appropriate σ and τ. If so, then we need only check (using oracle A) which of these strings is an initial segment of A.