Qualifying Exam
Logic
January 19, 2001

Instructions:

If you signed up for Computability Theory, do two E and two C problems.

If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Show that every countable ordinal has the same order type as a closed set of reals.

E2. Show that the set $\{\langle n, m, p\rangle \mid n + m = p\}$ is not definable in the structure $(\mathbb{N}, \cdot)$.

E3. Gödel's First Incompleteness Theorem tells us that there is a Π sentence φ such that φ is true in $(\mathbb{N}, +, \cdot, 0, 1)$, but is not provable in Peano arithmetic. Is there a Σ sentence with this property? If so, write it down, and if not, prove that no Σ sentence has this property.

Recall that a Π sentence is one of the form $\forall x \psi$, where ψ is quantifier free, and a Σ formula is one of the form $\exists x \psi$, where ψ is quantifier free.

C1. Prove that there exists x such that

$$\forall y(x \in W_y \Leftrightarrow y \in W_x).$$

C2. Let A be c.e. Prove that there is no total A-computable function f such that for all e, if W_e is finite then $W_e \subseteq \{0, \ldots, f(e)\}$.

C3. Show that there is a noncomputable c.e. set A such that for any disjoint c.e. sets U and V with $A = U \cup V$, if A is U-computable then V is computable.

Note: c.e. is the same as r.e., computable is the same as recursive, A is U-computable is the same as A is Turing reducible to U or $A \leq_T U$, and W_e is the e^{th} c.e. set in some standard enumeration.

S1. Let $[\mathbb{R}]^{<\omega}$ denote the set of all finite subsets of the reals and $[\mathbb{R}]^{\omega}$ denote the set of all countably infinite subsets of the reals. Prove that CH is equivalent to the following statement:

(P) There is a function $F : [\mathbb{R}]^{<\omega} \to [\mathbb{R}]^{\omega}$ such that for every $A \in [\mathbb{R}]^{<\omega}$, we have $a \in F(A \setminus \{a\})$ for all but at most one $a \in A$.

S2. Prove there exists a family $\mathcal{P}$ of perfect subtrees of $2^{<\omega}$ which when ordered by reverse inclusion is an ω_1-Aronszajn tree.

Notation. An ω_1-Aronszajn tree is a tree $\mathcal{P}$ of height ω_1 such that $\mathcal{P}$ has no uncountable branches and such that $\mathcal{P}_\alpha$ (the α^{th} level of $\mathcal{P}$) is countable for each α. A subtree $p \subseteq 2^{<\omega}$ is perfect iff every node in p has incompatible nodes above it.

Hint. You can construct $\mathcal{P}$ inductively, with the root equal $2^{<\omega}$. You have to make sure that $\mathcal{P}$ doesn't die at limit levels, so maintain the property that for any $\alpha < \beta$, $n < \omega$, and $p \in \mathcal{P}_\alpha$ there exists $q \in \mathcal{P}_\beta$ with $q \subseteq p$ and $q \cap 2^n = p \cap 2^n$.

S3. Prove that the following is consistent with $\neg CH$: There are cofinal $A_\gamma \subset \gamma$, for γ a countable limit ordinal, such that each A_γ has order type ω and such that whenever A is an unbounded subset of ω_1, there is a closed unbounded $C \subseteq \omega_1$ such that $A \cap A_\gamma$ is infinite for all γ in C.

Hint. Use Cohen forcing, and use the γ^{th} Cohen real to code A_γ.

ANSWERS

E1. Using induction on $\alpha < \omega_1$ and also that any two open intervals are order isomorphic. This can also be proved without using the axiom of choice.

E2. Take any permutation σ of the primes. It extends to an automorphism of $(\mathbb{N}, \cdot)$ using the fundamental theorem of arithmetic. But $<$ is definable from $+$ and $(\mathbb{N}, <)$ has no nontrivial automorphisms.

E3. Any Σ sentence true in $\mathbb{N}$ is provable in Peano arithmetic. All one has to prove is that all quantifier free sentences true in $\mathbb{N}$ are provable in PA. Let $\underline{n}$ stand for $1+1+1+\cdots+1$ n-times (or 0 if $n=0$). By induction one can prove that

$$PA \vdash \underline{n} + \underline{m} = \underline{n+m}$$

$$PA \vdash \underline{n} \cdot \underline{m} = \underline{nm}$$

$$n \neq m \text{ implies } PA \vdash \underline{n} \neq \underline{m}$$

This collection of sentences is known as Robinson's Q. It follows that any model of PA is a model of Q, hence any quantifier free sentence true in $\mathbb{N}$ is true in all models of PA.

S1. Assume CH. Replace $\mathbb{R}$ by ω_1, and let $F(A) = A \cup \bigcup A = A \cup \bigcup_{\alpha \in A} \alpha$.

Assume (P) and not CH. Take $C \subseteq \mathbb{R}$ of size ω_1. There exists $a \in \mathbb{R} \setminus \bigcup_{b \in C} F(\{b\})$. For each $b \in C$ we have $b \in F(\{a\})$, so $F(\{a\})$ is uncountable, contradicting (P).

S2. A key argument for constructing the limit levels $\mathcal{P}_\lambda$ is the fusion lemma: If $(p_n : n < \omega)$ are a descending sequence of perfect trees and $(k_n : n < \omega)$ is an increasing sequence such that for all n we have

$p_{n+1} \cap 2^{k_n} = p_n \cap 2^{k_n}$ and

all nodes in $p_{n+1} \cap 2^{k_n}$ have at least two extensions in $p_{n+1} \cap 2^{k_{n+1}}$,

then $\cap_{n<\omega} p_n$ is a perfect tree. Another property which should be in the construction is that any distinct p, q at the same level should have no infinite branches in common (or equivalently $p \cap q$ is finite). $\mathcal{P}$ has no ω_1 branch since there cannot be an ω_1 descending chain in the power set of $2^{<\omega}$.

S3. In the ground model, V, choose a map f_γ from ω onto γ for each countable limit γ. Let $V[G]$ add Cohen reals, $\{r_\alpha : \alpha < \kappa\} \subset \omega^\omega$. To ensure $\neg CH$, let $\kappa \geq \omega_2$ (or, assume $V \models \neg CH$). Use r_γ to construct A_γ; for example, let $A_\gamma = \{\xi_n^\gamma : n \in \omega\}$, where $\xi_0^\gamma = 0$, and ξ_{n+1}^γ is $f_\gamma(r_\gamma(n))$ if this is greater than $\max(f_\gamma(n), \xi_n^\gamma)$, and $\max(f_\gamma(n), \xi_n^\gamma + 1)$ otherwise.

C1. Prove that there exists x such that

$$\forall y(x \in W_y \Leftrightarrow y \in W_x).$$

Proof: Define a computable function f such that $W_{f(x)} = \{y \mid x \in W_y\}$, and apply the Fixed-Point Theorem to get an index x_0 with $W_{x_0} = W_{f(x_0)}$.

C2. Let A be c.e. Prove that there is no total A-computable function f such that for all e, if W_e is finite then $W_e \subseteq \{0, \dots, f(e)\}$.

Proof: "W_e is finite" is a Σ^0_2-complete property, whereas for any A-computable function f, "$W_e \subseteq \{0, \dots, f(e)\}$" is Π^0_2.

C3. Show that there is a noncomputable c.e. set A such that for any disjoint c.e. sets U and V, if $A = U \cup V$ and one of U or V computes A then the other is computable.

Proof by priority argument: Try to show V computable at x after U computes $A(x)$. If later x enters A (and thus may now enter V), then either x enters U (and thus not V), or x enters only V and so we can kill the reduction from U to A at x (as $A(x)$ is no longer correctly computed by U).

(Comment: This property is called mitotic and is known to be equivalent to autoredicibility, but this is not relevant here.)