Qualifying Exam
Logic
August 2002

Set Theory

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let κ be a cardinal with $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Prove that the following are equivalent:

1. For all $X \subseteq \mathbb{R}$ with $|X| = \kappa$, there is a $q \in \mathbb{Q}$ such that $|X \cap (-\infty, q)| = |X \cap (q, +\infty)| = \kappa$.
2. $\mathrm{cf}(\kappa) > \omega$.

E2. Let $\mathcal{L}$ have one binary relation $<$ and one unary relation U. Let $\mathfrak{R}$ be the model $(\mathbb{R}; < \mathbb{Q})$, where $<$ is the usual order; that is, U is interpreted as the subset $\mathbb{Q}$. Prove that the theory of $\mathfrak{R}$ is decidable.

E3. Prove that the theory of $(\mathbb{C}; +, \cdot, \exp)$ is undecidable. Here, exp denotes the exponential function, e^z.

The *theory* of a model is the set of sentences true in that model.

C1. Let g be computable. Show there exists e such that
(a) W_e is computable and
(b) if e' is the least such that $W_{e'} = \overline{W_e}$, then $e' > g(e)$.

C2. Prove or disprove: For every total computable function f, there exists e such that $W_{f(f(e))} = W_{f(e)}$.

C3. Prove or disprove: There are simple sets A and B such that $A \cap B$ is not simple.

W_a (for $a \in \omega$) denotes the a^{th} computably enumerable set (in some standard enumeration). $\overline{X}$ denotes $\omega \setminus X$. You can assume that W_a is the domain of the a^{th} partial computable function, φ_a.

Model Theory

M1. Let $\mathcal{L}$ be a first-order language, $\phi(x,y)$ an $\mathcal{L}$-formula and $\mathfrak{M}$ an ω_1-saturated $\mathcal{L}$-structure. Assume that there is a sequence $(a_i : i \in \mathbb{N})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(a_i, a_j) \quad \Longleftrightarrow \quad i < j \qquad \text{for all } i, j \in \mathbb{N}.$$

(a) Prove that there is a set $(b_i : i \in \mathbb{Q})$ of elements of $\mathfrak{M}$ such that

$$\mathfrak{M} \models \phi(b_i, b_j) \quad \Longleftrightarrow \quad i < j \qquad \text{for all } i, j \in \mathbb{Q}.$$

(b) Conclude that $\mathfrak{M}$ is not ω-stable, that is, there is a countable $B \subseteq M$ with uncountably many 1-types over B.

For the next problem, we need the following definitions: let $\mathcal{F}$ be the collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. We define an equivalence relation $\sim$ on $\mathcal{F}$ by

$$f \sim g \qquad \Longleftrightarrow \qquad \text{there is } a \in \mathbb{R} \text{ such that } f(x) = g(x) \text{ for all } x > a.$$

Given $f \in \mathcal{F}$, we denote by $[f]$ the equivalence class of f under $\sim$ (called the **germ** of f at $= \infty$), and we put $\mathcal{HH} = \mathcal{F}/\sim$. We also let $<$ be a partial ordering on $\mathcal{HH}$ defined by

$$[f] < [g] \qquad \Longleftrightarrow \qquad \text{there is } a \in \mathbb{R} \text{ such that } f(x) < g(x) \text{ for all } x > a.$$

M2. Let $\mathcal{R}$ be an expansion of the real field $(\mathbb{R}, <, +, -, \cdot, 0, 1)$, and put $\mathcal{HH}(\mathcal{R}) = \{[f] : f : \mathbb{R} \longrightarrow \mathbb{R} \text{ is definable in } \mathcal{R}\} \subseteq \mathcal{HH}$. Prove that $\mathcal{R}$ is o-minimal iff $\mathcal{HH}(\mathcal{R})$ is totally ordered by $<$. (Hint: use the Monotonicity Theorem for $\Longrightarrow$ and characteristic functions for $\Longleftarrow$.)

M3. Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ an infinite, strongly minimal $\mathcal{L}$-structure. Below we let $\Pi_{n-1} : M^n \longrightarrow M^{n-1}$ be the projection on the first $n-1$ coordinates, and for a set $S \subseteq M^n$ and $x' \in M^{n-1}$, we put $S_{x'} = \{x_n \in M : (x', x_n) \in S\}$.

We define by induction on $n \in \mathbb{N}$ what it means for a definable set $B \subseteq M^n$ to be **σ-finite**, where $\sigma \in \{0,1\}^{\{1,\ldots,n\}}$:

- if $n = 1$, then B is 0-finite iff B is finite and 1-finite iff B is cofinite;
- if $n > 1$ and $\sigma \in \{0,1\}^{\{1,\ldots,n\}}$, then B is σ-finite iff the set $\Pi_{n-1}(B)$ is σ'-finite, where $\sigma' \in \{0,1\}^{\{1,\ldots,n-1\}}$ is given by $\sigma'(i) = \sigma(i)$ for all $i = 1, \ldots, n-1$, and $B_{x'}$ is
  - finite for all $x' \in \Pi_{n-1}(B)$ if $\sigma(n) = 0$,
  - cofinite for all $x' \in \Pi_{n-1}(B)$ if $\sigma(n) = 1$.

(a) Let $B = \phi(M^n)$, where $n \in \mathbb{N}$ and $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula. Assume that $\sigma \in \{0,1\}^{\{1,\ldots,n\}}$ and B is σ-finite, and put $\mathrm{gdim}(B) = \sum_{i=0}^{n} \sigma(i)$. Prove that

$$\mathrm{gdim}(B) = \max\{\dim(a) : \ a \in \phi((M^*)^n), \ \mathcal{M} \preceq \mathcal{M}^*\},$$

where $\dim(a)$ is the dimension of the tuple a in the sense of the pregeometry defined on $\mathcal{M}$ by the (model-theoretic) algebraic closure operation.

(b) Let $A = \phi(M^n)$, where $n \in \mathbb{N}$ and $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula, and put $\mathrm{gdim}(A) = \max\{\mathrm{gdim}(B) : \ B \subseteq A \text{ and } B \text{ is } \sigma\text{-finite for some } \sigma\}$. Prove that

$$\mathrm{gdim}(A) = \max\{\dim(a) : \ a \in \phi((M^*)^n), \ \mathcal{M} \preceq \mathcal{M}^*\}.$$

S1. Call $B \subseteq \omega$ *big* iff $\{n, n+1, n+2, \ldots, 2n\} \subset B$ for infinitely many n. Assume $MA(\kappa)$, and let $\mathcal{F}$ be a family of big subsets of ω such that $\mathcal{F}$ is closed under finite intersections and $|\mathcal{F}| < \kappa$ Prove that there is a big set C such that $C \subseteq^* B$ for all $B \in \mathcal{F}$.

S2. Let κ be a regular uncountable cardinal and $\cdot$ a function from $\kappa \times \kappa$ into κ which makes $(\kappa, \cdot)$ a group, with identity element 1. Observe that there is a club (closed unbounded set) $C \subseteq \kappa$ such that $(\alpha, \cdot)$ is a group for all $\alpha \in C$. Now, assume that $(\kappa, \cdot)$ is simple. Prove that there is a club $C \subseteq \kappa$ such that $(\alpha, \cdot)$ is simple for all $\alpha \in C$.

Remark. The group G is simple iff it has no normal subgroups other than G or $\{1\}$, but for this problem, it may be better to say that G is *non*-simple iff there are elements $a, b \neq 1$ such that b is not a finite product of elements from $\{x^{-1}ax : x \in G\} \cup \{x^{-1}a^{-1}x : x \in G\}$

S3. Let M be a countable transitive model of ZFC. Let $\mathbb{P}$ be Cohen forcing, $Fn(\omega \times \omega, 2)$. Let G be $\mathbb{P}$-generic over M, and let $F = \bigcup G : \omega \times \omega \to 2$. Via binary expansion, F codes ω real numbers, $r_n \in [0, 1]$, where $r_n = \sum_{i \in \omega} F(\omega \cdot n + i) \cdot 2^{-i-1}$. In $M[G]$ we have $E = \{r_n : n \in \omega\}$. Prove that E is densely ordered; that is if $x, y \in E$ and $x < y$, then $x < z < y$ for some $z \in E$.

Answers

E1. If $\mathrm{cf}(\kappa) = \omega$ and $\lambda_n \nearrow \kappa$, then one can put a set of size λ_n in $(n, n+1)$ to get a counter-example. If $\mathrm{cf}(\kappa) > \omega$, let $A = \{q \in \mathbb{Q} : |X \cap (-\infty, q)| = \kappa\}$ and $B = \{q \in \mathbb{Q} : |X \cap (q, +\infty))| = \kappa\}$. Note that $A \cup B = \mathbb{Q}$. Also, $A \neq \emptyset$ and $B \neq \emptyset$ (by $\mathrm{cf}(\kappa) > \omega$). If $A \cap B \neq \emptyset$, we are done. If $A \cap B = \emptyset$, then $B < A$; let

using the fact that at least one of $X \cap (-\infty, z)$ and $X \cap (z, +\infty))$ has size κ.

E2. Let T be the theory which says that $<$ is a dense total order without endpoints and U is a dense subset, and prove that T is $\aleph_0$-categorical, and hence complete.

E3. Since the theory of $(\mathbb{Z}; +, \cdot)$ is well-known to be undecidable, it is sufficient to show how to define $\mathbb{Z}$ in the model $(\mathbb{C}; +, \cdot, \exp)$. Note that $0, 1, -1$ are all definable. Then, you can define $A := \{z : e^z = 1\} = \{2n\pi i : n \in \mathbb{Z}\}$, and then $\mathbb{Z}$ is $\{w : \forall u \in A[wu \in A]\}$.

C1. See Rogers' book, p.216. Make $W_{f(x)}$ be finite but intersect each nonempty W_i for $i \leq g(x)$.

C2. Fix a, b such that $W_a \neq W_b$. Let $f(x)$ be a when $x \neq a$ and let $f(a) = b$. If $e \neq a$, then $f(e) = a$ and $f(f(e)) = b$. If $e = a$, then $f(e) = b$ and $f(f(e)) = a$. In either case, $W_{f(f(e))} \neq W_{f(e)}$.

M1. By compactness, let $\mathfrak{B}$ be an elementary extension of $\mathfrak{M}$ containing elements b_i as in part (a). List $\mathbb{Q}$ as $\{i_n : n \in \omega\}$. Since $\mathfrak{M}$ is ω_1-saturated, we can inductively find elements $d_{i_n} \in M$ such that $(b_{i_0}, b_{i_1}, \ldots, b_{i_n})$ and $(d_{i_0}, d_{i_1}, \ldots, d_{i_n})$ realize the same type.

For part (b), if $B = \{d_i : i \in \mathbb{Q}\}$, then there is at least one 1-type over B corresponding to each proper Dedekind cut in $\mathbb{Q}$.

S1. It is a standard result that there is an infinite C such that $C \subseteq^* B$ for all $B \in \mathcal{F}$. The problem can be solved using the same partial order used to produce the infinite C. Or, for each $A \subseteq \omega$, let

$$\widehat{A} = \{n : \{n, n+1, n+2, \ldots, 2n\} \subset A\} \quad .$$

Get an infinite X such that $X \subseteq^* \widehat{B}$ for all $B \in \mathcal{F}$, and let

$$C = \bigcup_{n \in X} \{n, n+1, n+2, \ldots, 2n\} \quad .$$

S2. By a standard argument, there is a club D such that all $\alpha \in D$ are limit ordinals satisfying $(\alpha, \cdot) \prec (\kappa, \cdot)$. If the result fails, there is a stationary

$S \subseteq D$ such that for $\alpha \in S$, $(\alpha, \cdot)$ is not simple, so that there are ordinals $a_\alpha, b_\alpha < \alpha$ with b_α not a finite product of elements from

$$\{x^{-1}a_\alpha x : x < \alpha\} \cup \{x^{-1}a_\alpha^{-1}x : x < \alpha\} \quad .$$

By the pressing-down lemma, fix a stationary T and ordinals $a, b < \kappa$ such that $a_\alpha = a$ and $b_\alpha = b$ for all $\alpha \in T$. But then, since T is unbounded, b is not a finite product of elements from

$$\{x^{-1}ax : x < \kappa\} \cup \{x^{-1}a^{-1}x : x < \kappa\} \quad ,$$

contradicting the assumuption that $(\kappa, \cdot)$ is simple.

S3. One can do this directly, but it may be easier to show that E is dense in $\mathbb{R}$. To do this, it is sufficient to show that for each $a, b \in Q$ and each $p \in \mathbb{P}$, there is an $n \in \omega$ and a $q \leq p$ such that $q \Vdash \check{a} < \dot{r}_n < \check{b}$. Here, one can choose any n such that p does not mention r_n (that is, $\omega \cdot n + i) \notin \text{dom}(p)$ for all i).