Qualifying Exam
Logic
August 2003

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Prove that there are countable $C_\alpha \subseteq \mathbb{R}$ for $\alpha < 2^{\aleph_0}$ such that C_α and C_β are not isomorphic (with respect to the usual order on the real numbers) whenever $\alpha < \beta < 2^{\aleph_0}$.

E2. Let $\mathcal{L} = \{<, U\}$, where U is a unary predicate. Let T be the set of $\mathcal{L}$–sentences which says that $<$ is a dense total order without first or last elements and U is closed downward; i.e.:

$$\forall x, y[U(y) \wedge x < y \rightarrow U(x)] \quad .$$

a. Up to isomorphism, how many countable models does T have?
b. Prove that $\{\varphi : T \vdash \varphi\}$ is computable.

E3. If $f : \omega \rightarrow \omega$, then f^n denotes f applied n times; e.g., $f^3(0) = f(f(f(0)))$. Give an example of a (total) computable f such that $\{f^n(0) : n \in \omega\}$ is not computable.

Terminology. A set is *computable* iff its characteristic function is computable.

Computability Theory

C1. Prove that there is no (total) computable $f : \omega \times \omega \to \omega$ such that for all $x, y \in \omega$: If W_x and W_y are both finite, then $f(x, y) = 1$ iff $|W_x| = |W_y|$.

Note: If W_x and W_y are not both finite, then $f(x, y)$ can be anything. $W_e = \text{dom}(\varphi_e)$ is the e^{th} c.e. set.

C2. Prove that there exists a computable linear order, $(\omega, \triangleleft)$, isomorphic to $\omega + \omega^*$ such that the ω part and the ω^* part are not computable (as subsets of ω). Here, ω^* denotes the reverse order type of ω, and the type $\omega + \omega^*$ consists of an ω^* stacked on top of an ω. So, the ω part is the set of all $n \in \omega$ such that $\{m \in \omega : m \triangleleft n\}$ is finite.

C3. For $V \subseteq \omega \times \omega$ define $V_e = \{x : \langle e, x \rangle \in V\}$. Prove or disprove:

1. There exists a computably enumerable $V \subseteq \omega \times \omega$ such that $\{V_e : e \in \omega\}$ is the set of all computable sets.
2. There exists a computable $V \subseteq \omega \times \omega$ such that $\{V_e : e \in \omega\}$ is the set of all computable sets.

M1. Let $(F;+,\cdot,<)$ be an ordered field. Prove that $\cdot$ is not first-order definable in $(F;+,<)$. Here, "first-order definable" allows the use of a fixed finite list of elements of F as parameters.

Hint. Prove that the theory of ordered abelian divisible groups is model complete.

M2. Let T be a complete $\mathcal{L}$-theory with infinite models. Assume that $\mathcal{L}$ contains the symbol $<$ and that T contains the axioms that $<$ is a total order. Assume that $|\mathcal{L}| = \aleph_1$. Prove that T is not $\aleph_2$–categorical.

M3. Let $\mathcal{M} = (M;<,\dots)$ be an expansion of a dense linear order without endpoints, and assume that any $\mathcal{N}$ elementarily equivalent to $\mathcal{M}$ is o-minimal. Prove that every definable (with parameters) subset of M^2 is a finite union of definable cells.

Hint. To simplify things, you may use the weak version of the Monotonicity Theorem: if $f : (a,b) \to M$ is definable, where $a,b \in M \cup \{-\infty,+\infty\}$, then there are $a_0 = a < a_1 < \cdots < a_k < a_{k+1} = b$ such that for every $i \in \{0,\dots,k\}$, the restriction of f to the interval (a_i,a_{i+1}) is continuous.

Terminology. A cell in M is either a point or an open interval with endpoints in $M \cup \{-\infty,+\infty\}$. A set $C \subseteq M^2$ is a cell if its projection I on the first coordinate is a cell and there are definable, continuous $f,g : I \to M$ such that either:

a. $C = \{(x,y) : x \in I\,,\ y = f(x)\}$, or
b. $C = \{(x,y) : x \in I\,,\ y > f(x)\}$, or
c. $C = \{(x,y) : x \in I\,,\ y < f(x)\}$, or
d. $C = \{(x,y) : x \in I\,,\ g(x) < y < f(x)\}$ and $g(x) < f(x)$ for all $x \in I$.

A model M is o-minimal iff every definable with parameters subset of M is a finite union of cells.

Set Theory

S1. Assume $V = L$. Assume that $A \prec L(\omega_2)$ and that A is uncountable. Prove that $\omega_1 \subseteq A$.

S2. Assume MA + ¬CH. Fix $f_n \in \{0,1\}^{\omega_1}$ for $n \in \omega$. Prove that there is an infinite $S \subseteq \omega$ and a $g \in \{0,1\}^{\omega_1}$ such that $\langle f_n : n \in S \rangle$ converges to g, in the sense that for α, $\{n \in S : f_n(\alpha) \neq g(\alpha)\}$ is finite.

S3. $\mathcal{F} \subseteq \omega^\omega$ is a *dominating family* iff $\forall g \in \omega^\omega \, \exists f \in \mathcal{F}(g \leq f)$; here, $g \leq f$ means $g(x) \leq f(x)$ for all $x \in \omega$. Now, let M be a countable transitive model for ZFC, let $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$ (finite partial functions from ω to ω), and let G be $\mathbb{P}$–generic over M. Let $\mathcal{F} = \omega^\omega \cap M$. Prove that $\mathcal{F}$ is not a dominating family in $M[G]$.

Answers

E1. List $\mathcal{P}(\omega)$ as $\{E_\alpha : \alpha < 2^{\aleph_0}\}$, and let

$$C_\alpha = \bigcup_{n\in\omega}[3n, 3n+1] \cup \bigcup_{n\in E_\alpha}\{3n+2\} \quad .$$

Prove that there are countable for $\alpha < 2^{\aleph_0}$ such that C_α and C_β are not isomorphic (with respect to the usual order on the real numbers) whenever $\alpha < \beta < 2^{\aleph_0}$.

E2. Up to isomorphism, countable models of T are of the form $\mathfrak{A} = (\mathbb{Q}; <, U_\mathfrak{A})$, and there are exactly five possibilities:

1. $U_\mathfrak{A} = \emptyset$.
2. $U_\mathfrak{A} = \mathbb{Q}$.
3. $U_\mathfrak{A} = (-\infty, q)$ for some $q \in \mathbb{Q}$.
4. $U_\mathfrak{A} = (-\infty, q]$ for some $q \in \mathbb{Q}$.
5. $U_\mathfrak{A} = (-\infty, x)$ for some $x \in \mathbb{R}\backslash\mathbb{Q}$.

Each of these may be characterized by a first-order sentence, $\sigma_1, \ldots, \sigma_5$, and each $T \cup \{\sigma_i\}$ is $\aleph_0$–categorical, and hence decidable. Then, $T \vdash \varphi$ iff every $T \cup \{\sigma_i\} \vdash \varphi$.

E3. Let $g : \omega \to \omega$ be computable but ran(g) not computable. Assume also that $g(0) = 0$ and g is 1-1. Define $f(x) = 2^0 3^0 = 1$ unless x is of the form $2^i 3^j$ for some i, j. Then, define $f(2^i 3^j)$ to be $2^{g(j+1)} 3^0$ if $g(j) = i$ and $2^i 3^{j+1}$ otherwise. Then $2^i \in \{f^n(0) : n \in \omega\}$ iff $i \in$ ran(g).

C1. Assume we had such an f. Let $E \subset \omega$ be c.e. but not computable, with $0 \in E$. Then $A := \{(n, n) : n \in E\}$ is c.e., and using the s^m_n Theorem, there is a computable $g : \omega \to \omega$ such that $W_{g(n)} = A_n$ for all n, where $A_n = \{y : (n, y) \in A\}$. But then $|W_{g(n)}|$ is 1 for $n \in E$ and 0 for $n \notin E$, and $E = \{x : f(x, 0) = 1\}$ would be decidable.

C2. Construct a computable sequences $\trianglelefteq_s$ of binary relations on initial segments of ω as follows. Each $\trianglelefteq_s$ is a strict linear ordering. Each extends the next. In addition, we have $\trianglelefteq_s$-Dedekind cuts L_s and R_s, i.e., everything in L_s is $\trianglelefteq_s$ left of everything in R_s.

Stage s=2n: put $2n, 2n+1$ into the cut between L_s and R_s, i.e.,
$L_{s+1} = L_s \cup \{2n\}$
$R_{s+1} = R_s \cup \{2n+1\}$
$L_s \trianglelefteq_{s+1} 2n \trianglelefteq_{s+1} 2n+1 \trianglelefteq_{s+1} R_s$

Stage s=2n+1: search for the least $e < s$ if any exists so that
(a) $W_{e,s} \cap L_s = \emptyset$

(b) $W_{e,s}$ contains the $\trianglelefteq_s$-rightmost $2e+1$ elements of R_s.

Move the gap over so that R_{s+1} is the $\trianglelefteq_s$-rightmost $2e$ elements (i.e. now L_{s+1} meets W_e. If no e exists do nothing.

We claim now that the resulting order is isomorphic to $\omega + \omega^*$ and that the ω^* part is not W_e for any e. Note that the left hand cut only increases (in fact, is an infinite ce set) and since even stages only add things at its end it eventually grows up into an isomorphic copy of ω.

Suppose inductively that each W_e acts only finitely many times. So suppose we are at a stage where no $\hat{e} < e$ will ever act again. Since larger $\hat{e} > e$ cannot move the rightmost $2e+1$ elements out of R it must be that once they are in R they always remain in R unless W_e acts. It follows that W_e will act at most one more time, if it does then W_e meets L, if doesn't it must be that W_e fails to contain the last $2e+1$ elements of R. In either case the last $2e+1$ elements of R will never be disturbed again.

C3. (1) This is true using increasing enumerations. At stage s put $V_{e,s}$ to be the range of $\phi_{e,s}$ on the largest $n < s$ such that $\phi_{e,s}$ restricted to n is a strictly increasing total map. If ϕ_e is not a strictly increasing total function, then V_e is finite (and clearly we will get all finite sets this way). If it is, then V_e is computable, since it has an increasing enumeration. Since all infinite computable sets have a strictly increasing enumeration, we get all of them. (2) This is false by the usual diagonalization.

M1. Assume that $\cdot$ is first-order definable, using $p_1, \ldots, p_n$ as parameters. WLOG, $0 < p_1 < \cdots < p_n$. Taking an elementary extension of $(F; +, \cdot, <)$, we may assume, WLOG, that F is non-archimedean. Fix an infinitely large $a \in F$ with $a > p_n$, and let G be the set of all rational combinations of $a, p_1, \ldots, p_n$. Then $(G; +, <)$ is an ordered divisible abelian group, so $(G; +, <) \prec (F; +, <)$. Under the assumption that product is definable, the restriction of $\cdot$ to G would make $(G; +, \cdot, <)$ into an ordered field, which is impossible, since $a^2 \notin G$.

Proof of Hint: This is an exercise in Chang and Keisler.

If we expand the language to be $+$, $<$, $>$, $\leq$, $\geq$, 0, $-$ with obvious definitions, then the theory of ordered abelian divisible groups eliminates quantifiers. To see this, let $\exists x\ \psi(x)$ be a formula such that $\psi(x)$ is conjunction of atomic formulas of the form

$n_i x \leq y_i$, $n_i x \geq y_i$, $n_i x < y_i$, or $n_i x > y_i$

where each n_i is a positive integer and y_i is a term not involving x. We can reduce to this case by replacing $u \neq v$ by $u < v$ or $v < u$ and $u = v$ by $u \leq v$ and $v \leq u$, and $\neg u \leq v$ by $v < u$ and so forth.

Now since $nx < y$ iff $mnx < my$, and so forth, by taking the product of all the n_i we may assume that they are all the same. Since the groups are divisible we can replace n by 1. Finally to eliminate x replace $x < y_i$ and $x > y_j$ by $y_j < y_i$ and so forth for $\leq$, etc.

M2. First assume that there is no greatest element. Using a compactness argument, it is easy to prove that for any model M of size $\aleph_2$ there exist an elementary extension N of size $\aleph_2$ with some element of N strictly greater than all elements of M. By using elementary chains of length either ω or ω_1 we can get models of size $\aleph_2$ with cofinality ω and ω_1. Clearly they can't be isomorphic. If there is a last element, then either there is a greatest element with no immediate predescessor and only finitely many successors or every model ends in an ω^*. In either case we add new elements just below this tail segment and above all the old elements.

Another way to do this is to use the fact that there must be models of size $\aleph_2$ which realize only $\aleph_1$ types over every subset of size $\aleph_1$. However, for a theory with a total order, one can always construct models which fail to have this property.

M3. Let $A \subseteq M^2$ be definable, and consider the definable set

$$B := \{(x, y) : \ y \text{ is in the boundary of the fiber } A_x\} \quad .$$

By o-minimality, each fiber B_x is finite, so by the assumption and the compactness theorem, there is $k \in \mathbb{N}$ such that $|B_x| \leq k$ for all $x \in M$. Hence by o-minimality, M can be partitioned into finitely many cells such that if I is one of these cells, there is an $l \in \{0, \ldots, k\}$ such that $|B_x| = l$ for all $x \in I$. So there are definable functions $f_1, \ldots, f_l : I \to M$ such that $f_1(x) < \cdots < f_l(x)$ for all $x \in I$ and $f_i(x) \in B_x$ for all x and i. By the Monotonicity Theorem, after shrinking I if necessary we may assume that each f_i is continuous. Hence $A \cap (I \times M)$ is a finite union of cells, and since I was arbitrary, the claim follows.

S1. Let $M \cong A \prec L(\omega_2)$, with M transitive and $j : M \to A$ the isomorphism. Then $M = L(\delta)$ for some $\delta \geq \omega_1$. Fix $\alpha < \omega_1$, and then fix $R \subset \omega \times \omega$ such that R well-orders ω in type α. Then $R \in M$ by $V = L$, so $R = j(R) \in A$, so $\alpha \in A$ by $A \prec L(\omega_2)$.

S2. Let $\mathcal{U}$ be a non-principal ultrafilter on ω and define $g(\alpha)$ to be the $i \in \{0, 1\}$ such that $K_\alpha := \{n \in \omega : f_n(\alpha) = i\} \in \mathcal{U}$. Then, use $\mathrm{MA} + \neg\mathrm{CH}$ to get S so that $S \subseteq^* K_\alpha$ for all $\alpha < \omega_1$.

S3. Let τ be the $\mathbb{P}$–name for the generic function added. If $\mathcal{F}$ were a dominating family in $M[G]$, then in M we would have a p and an $f \in \omega^\omega$ such that $p \Vdash \tau \leq \check{f}$. Now, extending p to make $p(n) > f(n)$ for some n yields a contradiction.