Qualifying Exam
Logic
January 2004

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. If $\mathcal{F} \subseteq \lambda^\lambda$, say that $\mathcal{F}$ *covers* λ iff for all $\alpha, \beta < \lambda$ there is an $f \in \mathcal{F}$ such that either $f(\alpha) = \beta$ or $f(\beta) = \alpha$. Let κ be any infinite cardinal Prove that κ^+ can be covered by a family of κ many functions, but not by any family of fewer than κ functions.

E2. Let $\mathcal{L} = \{+, 0\}$. Let Σ be the theory of infinite abelian groups plus the axiom $\forall x[x + x = 0 \ \vee \ x + x + x = 0]$. Prove that Σ is decidable (i.e., there is an algorithm for deciding whether $\Sigma \vdash \varphi$).

E3. For the following, give either a proof or a counter-example. α, β denote ordinals:

a. $\forall \alpha, \beta > \omega^2 \, [\alpha + \beta = \beta + \alpha \ \rightarrow \ \alpha \cdot \beta = \beta \cdot \alpha]$.
b. $\forall \alpha, \beta > \omega^2 \, [\alpha \cdot \beta = \beta \cdot \alpha \ \rightarrow \ \alpha + \beta = \beta + \alpha]$.

Computability Theory

C1. Suppose we are given $x, y \in 2^\omega$, $d \in \omega$, and a partial computable $\psi : 2^{<\omega} \times d \to 2^{<\omega}$ such that for every $n < \omega$, there exists $i < d$ such that $\psi(y{\restriction}n, i) \downarrow= x{\restriction}n$. Prove that $x \leq_T y$. (Here $x{\restriction}n \in 2^n$ is the restriction of x to the domain n.)

C2. Let $A, B \subseteq \omega$. Say that $A \leq_{tt} B$ (A *is truth-table reducible to* B) if there is a computable function f such that $x \in A$ iff D_y is an initial segment of B for some $y \in D_{f(x)}$. (Here D_y is the finite set S with canonical index y, i.e., $y = \sum_{n \in S} 2^n$.)

Show that $A \leq_{tt} B$ iff there is a Turing functional Φ such that $A = \Phi(B)$ and $\Phi(X)$ is total for all oracles X.

C3. Prove that the 1-degrees do not form an upper semilattice as follows:

a. Construct two Turing-incomparable simple sets A and B.

b. Suppose that $\deg_1(A)$ and $\deg_1(B)$ have a least upper bound $\deg_1(D)$. Show that for some $z \notin D$, $A, B \leq_1 D \cup \{z\}$. (Hint: Using simplicity, first find $z, z' \notin D$ such that $A \leq_1 D \cup \{z\}$ and $B \leq_1 D \cup \{z'\}$.)

c. Show that $D \leq_1 A \oplus B$ and that therefore D is simple. (Here $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.)

d. Show that $D \cup \{z\} <_1 D$ for a contradiction.

M1. Prove that the following are equivalent for a cardinal κ:

a. $\kappa \geq 2^{\aleph_0}$.

b. Whenever $\mathcal{F}$ is a family of elementarily equivalent countable structures for a countable language, there is a model $\mathfrak{B}$ with $|\mathfrak{B}| = \kappa$ such that each $\mathfrak{A} \in \mathcal{F}$ is elementarily embeddable into $\mathfrak{B}$.

M2. Let $\mathcal{L} = \{+, -, \cdot, 0, 1\}$. Let Σ in $\mathcal{L}$ be the theory of algebraically closed fields of characteristic 0. Let $\mathcal{L}' = \mathcal{L} \cup \{U\}$ where U is 1-place. Let Σ' in $\mathcal{L}'$ be the theory of pairs of models of Σ, so a model of Σ' is an algebraically closed field of characteristic 0 in which U is an algebraically closed proper subfield. Prove that Σ' is complete and model-complete.

M3. Assume that $\mathcal{L}$ contains only predicate symbols. Say that a structure $\mathfrak{A}$ is *partitionable* iff it is the disjoint union of two substructures each of which is isomorphic to $\mathfrak{A}$. Prove or disprove: If $\mathfrak{A}$ is partitionable and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{B}$ is partitionable.

Here, $\mathfrak{A}$ is the *disjoint union* of $\mathfrak{A}_1, \mathfrak{A}_2$ iff $\mathfrak{A}_1, \mathfrak{A}_2$ are submodels of $\mathfrak{A}$ and A is the disjoint union of A_1, A_2.

Set Theory

S1. Let κ be an uncountable regular cardinal. Call $S \subseteq [\kappa]^\omega$ *stationary* iff for every $f : [\kappa]^{<\omega} \to \kappa$ there exists $a \in S$ with $f([a]^{<\omega}) \subseteq a$. Assume that S is stationary and $S = \bigcup_{i\in\omega} S_i$. Prove that some S_i is stationary.

$[\kappa]^\omega$ denotes the family of countably infinite subsets of κ, and $[\kappa]^{<\omega}$ denotes the family of finite subsets of κ.

S2. $ZFC - P$ denotes the axioms of ZFC with Power Set deleted. Let Σ be $ZFC - P$ plus the axiom that there is an uncountable ordinal. If $M \models \Sigma$, then $(\omega_1)^M$ denotes what M thinks is the smallest uncountable ordinal. Prove that there are α, β, γ with $\omega < \alpha < \beta < \gamma < \omega_1$ such that $L(\beta) \models \Sigma$, $L(\gamma) \models \Sigma$, and $\alpha = (\omega_1)^{L(\beta)} = (\omega_1)^{L(\gamma)}$.

S3. Let M be a countable transitive model for ZFC, let $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$ (finite partial functions from ω to ω), and let G be $\mathbb{P}$–generic over M. Let κ be an infinite cardinal of M. Prove that the following are equivalent:

a. $\mathrm{cf}(\kappa) \neq \omega$ in M.

b. Whenever $S \subseteq \kappa$ in $M[G]$, there is an $A \subseteq \kappa$ in M such that $|A| = \kappa$ in M and either $A \subseteq S$ or $A \cap S = \emptyset$.

Answers

E1. Let $g_\alpha : \kappa \to \alpha + 1$ be onto for each $\alpha < \kappa^+$. For each $\xi < \kappa$ define $f_\xi : \kappa^+ \to \kappa^+$ by $f_\xi(\alpha) = g_\alpha(\xi)$. Let $\mathcal{F} = \{f_\xi : \xi < \kappa\}$. Given $\alpha, \beta < \kappa^+$ suppose that $\alpha \leq \beta$. Choose ξ so that $g_\beta(\xi) = \alpha$. Then $f_\xi(\beta) = \alpha$.

Now suppose $|\mathcal{F}| < \kappa$. By a Löwenheim-Skolem argument find α with $\kappa < \alpha < \kappa^+$ so that for every $f \in \mathcal{F}$ and for every $\delta < \alpha$ we have that $f(\delta) < \alpha$. Since α has cardinality κ and $|\mathcal{F}| < \kappa$ there exists $\beta < \alpha$ with $\beta \neq f(\alpha)$ for each $f \in \mathcal{F}$. But then there is no $f \in \mathcal{F}$ with $f(\beta) = \alpha$ or $f(\alpha) = \beta$.

E2. Let E_n be the assertion that the group has exponent n — that is $\forall x[nx = 0]$. Note that the given axiom is equivalent to $E_2 \vee E_3$, since if x has order 2 and y has order 3, then $x + y$ has order 6.

Then $\Sigma \vdash \varphi$ iff $\Sigma \cup \{E_2\} \vdash \varphi$ and $\Sigma \cup \{E_n\} \vdash \varphi$.

But for prime n, $\Sigma \cup \{E_n\}$ is $\aleph_0$ – categorical, hence complete, and hence decidable.

To see that it is categorical in any infinite cardinality note that any model of E_n can be viewed as a vector space over the field of size n. Any two such vector spaces will be isomorphic if they have the same dimension.

E3. Both are false:

a. $\alpha = \omega^3, \quad \beta = \omega^3 \cdot 2,$
$\alpha + \beta = \beta + \alpha = \omega^3 \cdot 3, \quad \alpha \cdot \beta = \omega^6 \cdot 2, \quad \beta \cdot \alpha = \omega^6.$

b. $\alpha = \omega^3, \quad \beta = \omega^4,$
$\alpha \cdot \beta = \beta \cdot \alpha = \omega^7, \quad \alpha + \beta = \omega^4, \quad \beta + \alpha = \omega^4 + \omega^3.$

C1. For each n let $S_n = \{s \in 2^n : \exists i < d\ [\phi(y \restriction n, i) \downarrow = s]\}$. Let k be the largest integer such that there are infinitely many n with $|S_n| = k$. Then $k \geq 1$ (since $x \restriction n \in S_n$) and $k \leq d$. Note that there is an infinite sequence n_i, computable in y, such that $n_0 < n_1 < n_2 < \cdots$ and each $|S_{n_i}| = k$. Let T be the tree of all sequences s such that for each $n_i < |s|$ we have that $s \restriction n_i \in S_{n_i}$. The tree T is computable in y and has at most k infinite paths. Hence each of these infinite paths is isolated and is therefor computable in T, and hence in y.

C2.

($\Rightarrow$) Given the computable function f, define the Turing functional Φ as follows: For any argument x of Φ, compute $z = 1 + \max\limits_{y \in D_{f(x)}} D_y$.
It is now easy to see that one can define $\Phi^\sigma(x)$ for each $\sigma \in 2^z$.

($\Leftarrow$) Given x, find by compactness of 2^ω a number z such that $\Phi^\sigma(x)$ is defined for all $\sigma \in 2^z$. It is now easy to find the appropriate set $D_{f(x)}$.

C3.

a. A simple finite-injury priority argument mixing simplicity with Friedberg-Muchnik strategies.

b. Let $A, B \leq_1 D$ via computable 1–1 functions f and g, say. Suppose $\overline{D} \cup \mathrm{ran}(f) = \omega$. Then one can easily find a computable 1–1 function h witnessing $D \leq_1 A$, thus contradicting $B \not\leq_1 A$. So fix $z \notin D \cup \mathrm{ran}(f)$, thus $A \leq_1 D \cup \{z\}$ via f also. Similarly find $z' \notin D$ such that $B \leq D \cup \{z'\}$ via h', say. Now slightly modify h' to obtain $B \leq_1 D \cup \{z\}$.

c. Clearly $A, B \leq_1 A \oplus B$, and $A \oplus B$ must be simple, so $D \leq_1 A \oplus B$ must also be simple.

d. Since D contains an infinite computable subset S, argue that $D \cup \{z\} \leq_1 D$ by "shifting within S". If $D \leq_1 D \cup \{z\}$ via some computable 1–1 function k, say, then the infinite c.e. set $\{z, k(z), k(k(z)), \dots\}$ is in $\overline{D}$, contradicting the simplicity of D.

M1. For (a) $\to$ (b): Let Σ be the theory of the structures in $\mathcal{F}$, let $\mathfrak{B}_0$ be an $\aleph_1$–saturated model of Σ with $|\mathfrak{B}_0| = 2^{\aleph_0}$, and let $\mathfrak{B}$ be an elementary extension of $\mathfrak{B}_0$ of size κ.

For (b) $\to$ (a): Let $\mathfrak{A}_0 = (\omega; +, \cdot, 0, S)$. Let $\mathcal{F}$ be the family of all countable models which are elementarily equivalent to $\mathfrak{A}_0$. If each $\mathfrak{A} \in \mathcal{F}$ is elementarily embeddable into $\mathfrak{B}$, then $\mathfrak{B}$ realizes all types over $\{S^n(0) : n \in \omega\}$, so that $|\mathfrak{B}| \geq 2^{\aleph_0}$.

M2. Completeness follows from model-completeness because there is a model $\mathfrak{M} \models \Sigma$ which embeds into every model of Σ — namely, let $U_{\mathfrak{M}}$ be the algebraic numbers and let M have transcendence degree 1.

To prove model-completeness, it is sufficient to assume $\mathfrak{A} \subset \mathfrak{B}$ with $\mathfrak{A}, \mathfrak{B} \models \Sigma$, and prove that every existential sentence true in $\mathfrak{B}_A$ is true in $\mathfrak{A}_A$. This is a little easier if you assume (WLOG) that the pair $(\mathfrak{A}, \mathfrak{B})$ is $\aleph_1$–saturated. Note that there are two cases; $U_{\mathfrak{B}}$ may or may not equal $U_{\mathfrak{A}}$. Σ does not have quantifier-elimination; for example, the formula $\exists x(xy = z \wedge U(x))$ is not equivalent to a quantifier-free formula of $\mathcal{L}$.

M3. This is false. In the language with countable many unary predicates P_n let T be the (complete) theory which says that P_0 is everything, and each P_{n+1} is an infinite coinfinite subset of P_n. Let $\mathfrak{A}$ be the countable model of T where the intersection of the P_n is empty and $\mathfrak{B}$ be any model of T where the intersection of the P_n has size one.

S1. Suppose that no S_i is stationary. For each i, let $f_i : [\kappa]^{<\omega} \to \kappa$ be such that no $a \in S_i$ satisfies $f_i([a]^{<\omega}) \subseteq a$. Let φ map $\omega \backslash \{1\}$ onto $\omega \times \omega$ such that if $\varphi(n) = (\varphi_1(n), \varphi_2(n))$, then $\varphi_2(n) \leq n$. Define $g : [\kappa]^{<\omega} \to \kappa$ as follows: $g(\{\alpha\}) = \alpha + 1$. If $|s| = n \neq 1$, let $t = t_s \subseteq s$ be the first $\varphi_2(n)$ elements of s, and let $g(s) = f_{\varphi_1(n)}(t_s)$. Now, fix $a \in \bigcup_{i\in\omega} S_i$ such that $g([a]^{<\omega}) \subseteq a$, and fix i with $a \in S_i$. We show that $f_i([a]^{<\omega}) \subseteq a$, a contradiction.

Note that a has no largest element (by our definition of $g(\{\alpha\})$. Now, fix $s \in [a]^{<\omega}$ and let $j = |s|$. Fix $n \neq 1$ with $\varphi(n) = (i, j)$. Then $n \geq j$, so let $s' \supseteq s$ be such that $s' \in [a]^n$ and s is the first j elements of s'. Then $f_i(s) = g(s') \in a$.

S2. WLOG $V = L$ (otherwise, work in L). First get ξ with $\omega_1 < \xi < \omega_2$ and $L(\xi) \prec L(\omega_2)$. Then, let A be a countable elementary submodel of $L(\omega_2)$ with $\omega_1, \xi \in A$, let $L(\gamma)$ be its Mostowski collapse, with $i : L(\gamma) \to A$ the corresponding isomorphism. Then let $i(\alpha) = \omega_1$ and $i(\beta) = \xi$.

S3. For (a) $\to$ (b): Working in M: Say $p \Vdash \tau \subseteq \check{\kappa}$. Since $\mathrm{cf}(\kappa) \neq \omega$ and $\mathbb{P}$ is countable, there is a $q \leq p$ and an $A \subseteq \kappa$ such that $|A| = \kappa$ and either $\forall \alpha \in A\, [q \Vdash \check{\alpha} \in \tau]$ or $\forall \alpha \in A\, [q \Vdash \check{\alpha} \notin \tau]$.

For $\neg$(a) $\to \neg$(b): In M, let $\lambda_n \nearrow \kappa$. Let $c \in M[G]$ with $c \subset \omega$ and c Cohen generic over M; you just need that neither c nor $\omega \backslash c$ has an infinite subset in M. Let $S = \bigcup \{\lambda_{n+1} \backslash \lambda_n : n \in c\}$.