Qualifying Exam
Logic
August 2004

Instructions:

If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Suppose $\langle A_n : n \in \omega\rangle$ is a sequence of infinite sets. Prove there is a sequence $\langle B_n : n \in \omega\rangle$ such that
(1) $B_n \cap B_m = \emptyset$ for each $n \neq m$, and
(2) $B_n \subseteq A_n$ and $|B_n| = |A_n|$ for each n.

E2. Let $\mathcal{L} = \{E\}$, where E is binary. One may view $\mathcal{L}$-structures as directed graphs. Let Σ be a set of axioms which say:
(1) The universe is infinite.
(2) There are no (directed) paths of length two:

$$\neg\exists x\exists y\exists z[E(x,y) \wedge E(y,z)] \quad .$$

(3) Every node is on some path of length one:

$$\forall x\exists y[E(x,y) \vee E(y,x)] \quad .$$

(4) Every node has out-degree 0 or 1:

$$\neg\exists x\exists y\exists z[E(x,y) \wedge E(x,z) \wedge y \neq z] \quad .$$

(5) Every node has in-degree 0 or 2:

$$\forall x\big[\neg\exists y E(y,x) \vee$$
$$\exists y\exists z[E(y,x) \wedge E(z,x) \wedge y \neq z \wedge \forall u[E(u,x) \rightarrow u = y \vee u = z]]\big] \quad .$$

Prove that Σ is complete.

E3. Prove that every nonstandard model of true arithmetic has a submodel which is not a model of true arithmetic. Here $\mathcal{L} = \{0, S, +, \cdot, <\}$, and true arithmetic is the set of sentences true in the standard model.

Model Theory

M1. Let Σ be a theory in a countable $\mathcal{L}$ such that Σ has infinite models. Prove that for any infinite cardinal κ, there is a model $\mathfrak{A} \models \Sigma$ of size κ such that every definable $D \subseteq A$ is either finite or of size κ. Here, D is *definable* iff D is of the form $\{a \in A : \mathfrak{A} \models \varphi[a, p_1, \ldots, p_m]\}$ for some fixed $p_1, \ldots, p_m \in A$ and some $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_m)$.

M2. Let $\mathcal{L} = \{\equiv\}$ and let Σ_0 be the theory which says that the model is infinite and $\equiv$ is an equivalence relation.

1. Prove that Σ_0 is ω–stable; that is, whenever $\mathfrak{A} \models \Sigma_0$ and S is a countable subset of $\mathfrak{A}$, there are only countably many 1-types over S.
2. Prove that Σ_0 has $2^{\aleph_0}$ complete extensions.
3. Prove that Σ_0 has exactly $\aleph_0$ extensions which are $\aleph_1$–categorical.

M3. Let Σ be a complete theory in a countable $\mathcal{L}$ such that Σ has infinite models. Assume that Σ is not $\aleph_0$-categorical.

1. Prove that Σ has at least two (nonisomorphic) countable homogeneous models.
2. Prove that Ehrenfeucht's example has exactly two (nonisomorphic) countable homogeneous models.

Here, a countable $\mathfrak{A} \models \Sigma$ is *homogeneous* iff whenever $n < \omega$ and $\vec{a}, \vec{b}$ are n-tuples from A which realize the same n-type, there is an automorphism of $\mathfrak{A}$ which moves $\vec{a}$ to $\vec{b}$. In *Ehrenfeucht's example*, $\mathcal{L} = \{<\} \cup \{c_i : i < \omega\}$, and Σ is the theory of dense total orders without endpoints (expressed just using $<$), plus $c_0 < c_1 < c_2 < \cdots$.

S1. Call $f \in \omega^\omega$ *big* iff $f(n) \neq 0$ for all n and $\sum_{n\in\omega} \frac{1}{f(n)} < \infty$. So, $n \mapsto n+1$ isn't big, but $n \mapsto 2^n$ is big. Assume $MA(\kappa)$ and let f_α be big for all $\alpha < \kappa$. Prove that there is a big g such that $g \leq^* f_\alpha$ for all α. Here, $g \leq^* h$ means that $g(n) \leq h(n)$ for all but finitely many n.

S2. Given an infinite family $\mathcal{F}$ of finite sets of reals so that

$$\sup_{A\in\mathcal{F}} |A| = \infty \quad ,$$

prove that there exists an infinite sequence $\langle I_n : n \in \omega \rangle$ of pairwise disjoint open intervals in the reals so that for each n, there is at least one $A \in \mathcal{F}$ such that both endpoints of I_n are in A.

S3. Let M be a countable transitive model for ZFC. Let $\mathbb{P}$ be Cohen forcing — that is, finite partial functions from ω to 2. Let G be $\mathbb{P}$-generic over M. Prove that in $M[G]$, there is a sequence of rationals, $\langle q_n : n \in \omega \rangle$ such that each real in $\mathbb{R} \cap M$ is in $\bigcup_{n<\omega}(q_n, q_n + 2^{-n!})$.

Answers

E1. Let $\kappa_n = |A_n|$ and $\lambda = \sup_n \kappa_n$. Fix $f : \lambda \to \omega$ such that each $f^{-1}\{n\}$ has size κ_n and is a subset of κ_n. Choose $b_\alpha \in A_{f(\alpha)} \setminus \{b_\xi : \xi < \alpha\}$ for $\alpha < \lambda$. This is possible because if $f(\alpha) = n$ then $\alpha < \kappa_n$. Then let $B_n = \{b_\alpha : f(\alpha) = n\}$.

To obtain such an f let $\{(\beta, n) : n < \omega,\ \beta < \kappa_n\} = \{(\beta_\alpha, n_\alpha) : \alpha < \lambda\}$ be listed in max-lex order, this is the same as lexicographical order on this set except that $\omega \times \omega$ is an initial segment of order type ω. Put $f(\alpha) = n_\alpha$.

Alternative solution: Suppose that $| \cup_n A_n| = \lambda$. Construct sequences $(B_n^\alpha : n < \omega)$ by induction on $\alpha < \lambda$ such that

- $|B_n^0| = \omega$
- $B_n^\alpha \subseteq A_n$
- $B_n^\alpha \subseteq B_n^\beta$ whenever $\alpha \leq \beta$
- B_n^α and B_m^α are disjoint unless $n = m$,
- $B_n^\delta = \cup_{\alpha<\delta} B_n^\alpha$ at limit ordinals δ
- $B_n^{\alpha+1} = B_n^\alpha$ if $|A_n| \leq |\alpha| + \omega$ and
- $|B_n^{\alpha+1} \setminus B_n^\alpha| = 1$ if $|A_n| > |\alpha| + \omega$

The first item is achieved as follows. Choose $b_n \in A_i \setminus \{b_m : m < n\}$ where $n = 2^i(2j+1)$ inductively on n. Then put

$$B_i^0 = \{b_n : \exists j\ n = 2^i(2j+1)\}.$$

The last item is possible to achieve since for these n we have that

$$|A_n| > |\alpha| + \omega = | \cup_m B_m^\alpha|$$

so we may inductively pick one new element for each such n.

Finally set $B_n = \cup_{\alpha<\kappa} B_n^\alpha$.

E2. By (1), it is enough to show that Σ is $\aleph_0$-categorical. Let $\mathfrak{A} \models \Sigma$. By (2)(3), we can partition A into P, Q, where $P = \{x : \exists y E(x, y)\}$ and $Q = \{x : \exists y E(y, x)\}$. By (3), $E_\mathfrak{A}$ contains no pairs in $P \times P\ \cup\ Q \times Q$, so $E_\mathfrak{A} \subseteq P \times Q$. By (4), $E_\mathfrak{A}$ is a function, and then (5) says that each element of Q has exactly 2 preimages. Hence, Σ is κ-categorical for all infinite κ.

E3. Let $\mathfrak{A}$ be a nonstandard model, and let H be any nonstandard element of $\mathfrak{A}$ and let $B = \{a \in A : a \text{ is standard or } H < a\}$.

Alternate solution which gives an initial segment: Let $P \in A$ be an infinitely large prime, and let $B = \{a \in A : \exists k \in \omega\,[a < P^k]\}$. Then the statement $\forall x\, \exists y > x\, \forall z\, [z < x \to z \mid y]$, which is true in the standard model, is false in $\mathfrak{B}$ (consider $x = 2P$).

M1. Let $\mathfrak{B} \models \Sigma$ with $|\mathfrak{B}| = \kappa$. Construct an elementary chain $\mathfrak{B} = \mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \cdots \prec \mathfrak{B}_\kappa$, as in the construction of a saturated model. $\mathfrak{A} = \mathfrak{B}_\kappa$. All the $\mathfrak{B}_\alpha$ will have size κ. Take unions at limit ordinals. By the usual bookkeeping, make sure that all infinite definable sets get expanded κ times.

M2.

1. It is enough to show that every $\mathfrak{A} \models \Sigma_0$ only realizes countably many 1-types over a countable S. Given $\mathfrak{A} \models \Sigma_0$, define $a \sim b$ iff there is an automorphism of $\mathfrak{A}$ taking a to b which fixes all elements of S. By taking elementary extensions, it is sufficient to consider only $\mathfrak{A}$ in which all infinite $\equiv$-classes have the same size. For these $\mathfrak{A}$ and countable $S \subseteq \mathfrak{A}$, there are only countably many $\sim$-classes in A.

2. For each $f : \omega \to \omega$, there is an extension which says that there are exactly $f(n)$ equivalence classes of size n.

3. For each n, the theory with exactly one class of sizes $1, 2, \ldots, n$ plus one infinite class is $\aleph_1$-categorical. This yields infinitely many $\aleph_1$-categorical extensions. To prove that there are only countably many, say $\Sigma \supseteq \Sigma_0$ is $\aleph_1$–categorical. Then every $\mathfrak{A} \models \Sigma$ can have at most 1 infinite equivalence class; otherwise there would be $\mathfrak{A}, \mathfrak{B} \models \Sigma$ with $\mathfrak{A} \prec \mathfrak{B}$ and $|\mathfrak{A}| = |\mathfrak{B}| = \aleph_1$, where all infinite classes have size $\aleph_1$ in $\mathfrak{B}$ but some infinite class has size $\aleph_0$ in $\mathfrak{A}$. Then, by compactness, there is a fixed n so that models of Σ have no equivalence classes of any finite size larger than n. There's only countably many possibilities for such Σ.

M3. If for some n, Σ has uncountably many n-types, then there are $2^{\aleph_0}$ countable homogeneous models, since each type is realized in some countable homogeneous model. If for all n, there are only countably many n-types, then there is a saturated model and an atomic model; these are both homogeneous, and they are not isomorphic because Σ is not $\aleph_0$-categorical. In Ehrenfeucht's example, the model in which $a = \sup_n c_n$ exists is not homogeneous, since a has the same type as all larger elements, but all automorphisms fix a.

S1. Let

$$\mathbb{P} = \{(n, g) \ : \ n < \omega \ \& \ g \in (\omega\backslash\{0\})^\omega \ \& \ \sum_i \frac{1}{g(i)} < 1\} \ .$$

and define $(n, g) \leq (m, h)$ iff $n \geq m$, $g{\restriction}m = h{\restriction}m$, and $g(i) \leq h(i)$ for all i. It is easy to see that the generic $r \in \omega^\omega$ satisfies $\sum 1/r(i) \leq 1$, so r is big. Given a big f define

$$D_f = \{(n, g) \in \mathbb{P} \ : \ n < \omega \ \& \ \forall i \geq n \ g(i) \leq f(i)\} \ .$$

To prove D_f is dense in $\mathbb{P}$: Since given any $(m, h) \in \mathbb{P}$ choose $\varepsilon > 0$ such that $\sum_i 1/h(i) < 1 - \varepsilon$ and then choose n sufficiently large so that $\sum_{i>n} 1/f(i) < \varepsilon/2$ and let g be defined by $g{\restriction}n = h{\restriction}n$ and $g(i) = \min(f(i), h(i))$ for $i \geq n$. Then $(n, g) \in D_f$.

Finally, to prove that $\mathbb{P}$ has the ccc: Suppose that $(n_\alpha, g_\alpha) \in \mathbb{P}$ for $\alpha < \omega_1$. Choose $\varepsilon > 0$ so that uncountably many of these, say $\alpha \in \Gamma$ have $\sum_i 1/g_\alpha(i) < 1 - \varepsilon$. Then for each of these choose m_α so that $\sum_{i>m_\alpha} 1/g_\alpha(i) < \varepsilon/2$ and then choose uncountable $\Delta \subseteq \Gamma$ so that all α in Δ have the same $m_\alpha = m$ and $g_\alpha{\restriction}m$. Then any two of these are compatible by just taking (n, h) where $h(i) = \min(g_\alpha(i), g_\beta(i))$.

S2. Inductively choose $(I_i : i < n), U_n$ so that

(1) $I_0, I_1, \ldots, I_{n-1}, U_n$ are pairwise disjoint open intervals with U_n possibly unbounded,

(2) for each $i < n$ there exists $A \in F$ with the endpoints of I_i in A, and

(3) $\sup_{A \in \mathcal{F}} |A \cap U_n| = \infty$.

To start we take $U_0 = (-\infty, \infty)$. To do the construction, choose $A \in F$ such that $A \cap U_n$ contains at least three elements $a < b < c$. Take $U_{n+1} = (b, \infty) \cap U_n$ or $U_{n+1} = (-\infty, b) \cap U_n$ whichever satisfies condition (3) and in the first case let $I_n = (a, b)$ and in the second case let $I_n = (b, c)$.

S3. In $M[G]$, let $F = \bigcup G$, so $F : \omega \to 2$. Let

$$q_n = \frac{|\{j < n : F(j) = 1\}|}{|\{j < n : F(j) = 0\}|} \ .$$

Then if $x \in \mathbb{R} \cap M$, each $p \in \mathbb{P}$ has an extension forcing $q_n < x < q_n + 2^{-n!}$ for some n. *Remark.* Many other methods of using an $F \in 2^\omega$ to code an ω-sequence of rationals will also work here.