Qualifying Exam
Logic
January 2007

Instructions:

If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. PA denotes Peano Arithmetic. If $\mathfrak{A} \models \mathrm{PA}$, let $\mathrm{pr}(\mathfrak{A})$ be the set of all $a \in \mathfrak{A}$ such that $\mathfrak{A} \models$ "a is prime". Prove:

a. For any $\mathfrak{A} \models \mathrm{PA}$, $|\mathrm{pr}(\mathfrak{A})| = |\mathfrak{A}|$.

b. For any $\mathfrak{A} \models \mathrm{PA}$ and $S \subseteq \mathrm{pr}(\mathfrak{A})$, there is a $\mathfrak{B} \succcurlyeq \mathfrak{A}$ with a $b \in B$ such that for all $a \in \mathrm{pr}(\mathfrak{A})$: $\mathfrak{B} \models a \mid b$ iff $a \in S$.

c. If Σ is a complete extension of PA, then Σ has exactly $2^{\aleph_0}$ non-isomorphic countable models.

E2. If $(X, <)$ is totally ordered set, let $I(X, <)$ be the set of strictly increasing functions $f : X \to X$ (that is $f(x) < f(y)$ whenever $x < y$).

a. Prove that $|I(\mathbb{R}, <)| = 2^{\aleph_0}$ (where $<$ is the usual order).

b. Give an example of a total order $(X, <)$ with $|X| = 2^{\aleph_0}$ and $|I(X, <)| = 2^{2^{\aleph_0}}$.

E3. Work in ZF (without AC), but assume that for every non-empty set X, there is a function $\bullet : X \times X \to X$ which makes X a group. Prove that AC holds. *Hint.* To well-order X, get a group operation on $X \cup \kappa$ for a suitably large κ.

M1. Let $\mathfrak{A}$ be a countable proper elementary extension of $(\omega; +, \cdot, <)$. Call $E \subseteq A$ *nice* iff E is an initial segment of A and E is closed under $+$ and $\cdot$ (so E is a sub-model of A, but not necessarily elementary). For example, ω and A are nice subsets of A. Prove:

a. If $a \in A$, then there is a nice $E \subseteq A$ such that $a \in E \neq A$.
b. There are $2^{\aleph_0}$ nice subsets of A.

M2. Let $\text{sat}_n(\mathfrak{A}, \mathfrak{B})$ abbreviate the statement that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ and every n–type over any countable subset of A is realized in B. Then:

a. Prove that $\text{sat}_m(\mathfrak{A}, \mathfrak{B})$ and $\text{sat}_n(\mathfrak{B}, \mathfrak{C})$ implies $\text{sat}_{m+n}(\mathfrak{A}, \mathfrak{C})$.
b. Give an example where $\text{sat}_1(\mathfrak{A}, \mathfrak{B})$ is true and $\text{sat}_2(\mathfrak{A}, \mathfrak{B})$ is false.

M3. Let $\mathfrak{R} = (\mathbb{R}; +, \cdot, <)$. Note that this structure is o-minimal; that is, every subset of $\mathbb{R}$ which is definable in $\mathfrak{R}$ is a finite union of points and intervals. Here, "*definable*" means definable by a formula, using a finite list of parameters from $\mathbb{R}$; *intervals* are of the form (a, b), where $a, b \in \mathbb{R} \cup \{\pm\infty\}$.

Now, let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\mathfrak{R}$. Prove that f is piecewise monotonic and piecewise continuous; that is, there are finitely many open intervals which cover all of $\mathbb{R}$ except for finitely many points such that f is monotonic (strictly increasing, strictly decreasing, or constant) and continuous on each interval.

Set Theory

S1. Assume $V = L$. Prove that for each $n < \omega$, there are ordinals $\alpha_0 < \alpha_1 < \cdots \alpha_n < \omega_1$ such that $L(\alpha_i) \prec L(\alpha_{i+1})$ for each $i < n$, but there is no $\beta > \alpha_n$ such that $L(\alpha_n) \prec L(\beta)$.

S2. Assume MA + $\neg$CH. Let S be a dense set of real numbers with $|S| = \aleph_1$. For $x \in S$, let $C_x \subseteq S \backslash \{x\}$ be a countable set which forms a simple sequence converging to x. Prove that there is an uncountable $T \subseteq S$ such that $T \cap C_x = \emptyset$ for each $x \in T$.

S3. Let M be a countable transitive model for ZFC, In M, let $\mathbb{P}$ be the partial order consisting of all finite partial functions from $\mathbb{Q}$ to $\mathbb{Q}$ which are both 1-1 and order-preserving. Let G be $\mathbb{P}$–generic over M. In $M[G]$, let $f = \bigcup G$. Note that M and $M[G]$ have the same $\mathbb{Q}$, but $M[G]$ has more reals (which you may view as Dedekind cuts in $\mathbb{Q}$). Prove that the following hold in $M[G]$:

a. f is an order-preserving bijection from $\mathbb{Q}$ onto $\mathbb{Q}$, and hence defines a continuous bijection $\widehat{f}$ from $\mathbb{R}$ onto $\mathbb{R}$.

b. If r is a real in M, then $\widehat{f}$ is not differentiable at r.

Answers

E1. For (a): Working in PA, one can define, by some formula $\varphi(x, y)$, the function which lists the primes in increasing order. If f is the the function in $\mathfrak{A}$ defined by φ, then f is a bijection from A onto $\mathrm{pr}(\mathfrak{A})$.

For (b): Let $\mathfrak{B} \models \Sigma$, where Σ is the elementary diagram of $\mathfrak{A}$ (written using $\mathcal{L} \cup \{c_a : a \in A\}$), together with the sentences $c_a \mid d$ for $a \in S$ and $c_a \nmid d$ for $a \in \mathrm{pr}(\mathfrak{A}) \backslash S$.

For (c): If P is the set of all standard primes, $S \subseteq P$, and $\mathfrak{A} \models \mathrm{PA}$, say that S is *coded by* $\mathfrak{A}$ iff for some $b \in A$, $S = \{p \in P : \mathfrak{A} \models p \mid b\}$. Then each S is coded by some countable $\mathfrak{A} \models \Sigma$, and each countable $\mathfrak{A}$ can code only countably many sets, so there must be $2^{\aleph_0}$ nonisomorphic models.

E2. For (a): The maps $x \mapsto x + c$, for $c \in \mathbb{R}$, show that $|I(\mathbb{R}, <)| \geq 2^{\aleph_0}$. To prove $|I(\mathbb{R}, <)| \leq 2^{\aleph_0}$: For $f \in I(\mathbb{R}, <)$ and $x \in \mathbb{R}$, let $f^+(x) = \lim_{t \to x+} f(t)$ and let $f^-(x) = \lim_{t \to x-} f(t)$. Let $J(f) = \{x \in \mathbb{R} : f^-(x) \neq f^+(x)\}$. Note that $J(f)$ is countable, and that each $f \in I(\mathbb{R}, <)$ is determined by the set $J(f)$ and the function $f \restriction (\mathbb{Q} \cup J(f))$.

For (b): Let $X = \mathbb{R} \times \mathbb{R}$, ordered lexicographically. So, X consists of $\mathbb{R}$ blocks, $\{a\} \times \mathbb{R}$, of order type $\mathbb{R}$. There are $2^{2^{\aleph_0}}$ order-automorphisms of X because for each $S \subseteq \mathbb{R}$, there is an order-automorphisms f such that for all $a \in \mathbb{R}$: $f(\{a\} \times \mathbb{R}) = \{a\} \times \mathbb{R}$ and $f(a, 0) = (a, 0) \leftrightarrow a \in S$.

E3. To well-order X: Let $\kappa = \aleph(X)$ (the Hartogs $\aleph$ function); so there is no injection from κ into X. WLOG, X contains no ordinals, so $\kappa \cap X = \emptyset$. Let $\bullet$ be a group operation on $\kappa \cup X$.

First note that $\forall z \in X\, \exists \alpha \in \kappa\, [z\alpha \in \kappa]$, because the map $\alpha \mapsto z\alpha$ is an injection, and the range of this injection cannot be contained in X.

Thus, for each $z \in X$, there are $\alpha, \beta \in \kappa$ such that $z\alpha = \beta$, or $z = \beta\alpha^{-1}$. Let $W = \{(\alpha, \beta) \in \kappa \times \kappa : \beta\alpha^{-1} \in X\}$. Then W maps onto X (by the map $(\alpha, \beta) \mapsto \beta\alpha^{-1}$), and W can be well-ordered (by lexicographic order), so X can be well-ordered.

M1. On $A\backslash\omega$, define $x \sim y$ iff there is an $n \in \omega$ such that $x^n > y$ and $y^n > x$. Observe that $\sim$ is an equivalence relation and the set of equivalence classes of $A\backslash\omega$ is densely ordered (and hence isomorphic to $\mathbb{Q}$). Let $[a]$ denote the equivalence class of a.

For (a), let $E = \omega \cup \bigcup\{[b] : b \in A\backslash\omega \ \& \ b \leq a\}$. For (b), note that every Dedekind cut in $A/\!\sim$ defines a nice subset of A, so that there is a nice subset of A for each real number.

M2. For (a): Observe that realizing types is is equivalent to realizing consistent sets of formulas of $\mathcal{L}_A$. We have $\mathfrak{A} \preccurlyeq \mathfrak{B} \preccurlyeq \mathfrak{C}$. Let $\Sigma(\vec{x}, \vec{y})$ be a consistent set of formulas over some countable $E \subseteq A$, where $\vec{x}$ denotes an m–tuple and $\vec{y}$ denotes an n–tuple. So, every finite subset of Σ is realized in $\mathfrak{A}$. We must show that $\Sigma(\vec{x}, \vec{y})$ is realized in $\mathfrak{C}$. For $\varphi(\vec{x}, \vec{y}) \in \Sigma(\vec{x}, \vec{y})$, let $\varphi'(\vec{x})$ be $\exists\vec{y}\varphi(\vec{x}, \vec{y})$, and let $\Sigma' = \{\varphi' : \varphi \in \Sigma\}$. Then $\Sigma'(\vec{x})$ is a consistent set of formulas over E in m variables, so is realized by some $\vec{b} \in B$ (using $\mathrm{sat}_m(\mathfrak{A}, \mathfrak{B})$). Now, replacing $\vec{x}$ by $\vec{b}$, we get $\Sigma^*(\vec{y}) = \Sigma(\vec{b}, \vec{y})$. WLOG, the original Σ was closed under $\wedge$, which implies that every finite subset of Σ^* is realized in $\mathfrak{B}$; thus Σ^* is a consistent set of formulas over the countable $E \cup \{\vec{b}\}$. Then Σ^* is realized by some $\vec{c} \in C$ (using $\mathrm{sat}_n(\mathfrak{B}, \mathfrak{C})$), and $(\vec{b}, \vec{c})$ realizes Σ.

For (b), let $\mathfrak{A}$ be the field of algebraic numbers and let $\mathfrak{B}$ be an algebraically closed field of transcendence degree 1. Then $\mathfrak{B}$ omits the two-type (over $\emptyset$) consisting of all $p(x, y) \neq 0$ such that p is a non-trivial polynomial over $\mathbb{Q}$.

M3. See Lou van den Dries, *Tame topology and o-minimal structures*, Chapter 3, Theorem 1.2, for a proof.

S1. We get $\alpha_0 < \alpha_1 < \cdots \alpha_n < \omega_1$ such that

$$L(\alpha_0) \prec L(\alpha_1) \prec \cdots \prec L(\alpha_n) \prec L(\omega_1) \qquad (*)$$

by the Löwenheim–Skolem–Tarski Theorem. Now, let α_n be the least ordinal such that $L(\alpha_n)$ is a model for ZF − P and such that there exist $\alpha_0 < \alpha_1 < \cdots \alpha_{n-1} < \alpha_n$ satisfying

$$L(\alpha_0) \prec L(\alpha_1) \prec \cdots \prec L(\alpha_n) \;\; ; \qquad (\dagger)$$

there is such an $\alpha_n < \omega_1$ by $(*)$. Let φ be the sentence which asserts the existence of a sequence of $n+1$ ordinals $\alpha_0 < \alpha_1 < \cdots \alpha_{n-1} < \alpha_n$ satisfying $(\dagger)$. Then $L(\alpha_n) \models \neg(\dagger)$ (because α_n is least), while $\beta > \alpha_n$ and $L(\beta) \models$ ZF −P implies that $L(\beta) \models (\dagger)$, so $L(\alpha_n) \not\prec L(\beta)$.

The "ZF − P" is included just to ensure that elementary model theory is absolute for the transitive models in question.

S2. Let $\mathbb{P}$ be the set of all finite $p \subseteq S$ such that $p \cap C_x = \emptyset$ for each $x \in p$. Define $q \leq p$ iff $q \supseteq p$. Note that $\emptyset = \mathbb{1} \in \mathbb{P}$ and $\{x\} \in \mathbb{P}$ for each $x \in S$. *If* $\mathbb{P}$ is ccc, then by $MA(\aleph_1)$, there are distinct x_α, for $\alpha < \omega_1$, so that the forcing conditions $\{x_\alpha\}$ are all compatible, and we can let $T = \{x_\alpha : \alpha < \omega_1\}$.

Now, suppose that A is an uncountable antichain in $\mathbb{P}$. We may assume that A forms a Δ system, and then, subtracting off the root, we may assume that the elements of A are pairwise disjoint. For $p \in A$, let $C(p) = \bigcup\{C_x : x \in p\}$. We may then list a subset of A as $\{p_\alpha : \alpha < \omega_1\}$ so that $p_\alpha \cap C(p_\xi) = \emptyset$ whenever $\xi < \alpha$. Then $p_\xi \cap C(p_\alpha) \neq \emptyset$ whenever $\xi < \alpha$, since $p_\alpha \perp p_\xi$. Since the p_α are disjoint, we may find open $U, V \subseteq \mathbb{R}$ such that $\overline{U} \cap \overline{V} = \emptyset$, and $p_\xi \subseteq U$ for uncountably many ξ and $p_\alpha \subseteq V$ for uncountably many α. Then, fix any α such that $p_\alpha \subseteq V$ and $p_\xi \subseteq U$ for infinitely many $\xi < \alpha$. Since $p_\xi \cap C(p_\alpha) \neq \emptyset$ for these ξ, we have $U \cap C(p_\alpha)$ is infinite, which is impossible, since the C_x for $x \in p_\alpha$ all converge to points in p_α, which is disjoint from $\overline{U}$.

S3. For (a): f is a union of a compatible family of order-preserving injections, so f is an order-preserving injection. $\{p : a \in \mathrm{dom}(p)\}$ and $\{p : a \in \mathrm{ran}(p)\}$ are dense and in M for each $a \in \mathbb{Q}$, so that $\mathrm{dom}(f) = \mathrm{ran}(f) = \mathbb{Q}$.

For (b), suppose that $\widehat{f}'(r) < n$, for some $n \in \omega$. Then there are $a, b \in \mathbb{Q}$ with $a < r < b$ such that whenever $a < c < r < d < b$: $\widehat{f}(d) - \widehat{f}(r) \leq n(d-r)$, and $\widehat{f}(r) - \widehat{f}(c) \leq n(r-c)$, so that $\widehat{f}(d) - \widehat{f}(c) \leq n(d-c)$. But

$$\{p \in \mathbb{P} : \exists c, d \in \mathrm{dom}(p)\; [a < c < r < d < b \;\&\; p(d) - p(c) > n(d-c)]\}$$

is in M (since $r \in M$) and is dense, a contradiction.