Qualifying Exam
Logic
January 2008

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let β be an ordinal. Assume that $\beta = X \cup Y$ and that X, Y both have order type $\alpha > 0$.

a. Prove that $\beta < \alpha + \alpha + \alpha$.
b. Give an example where $\alpha + \alpha < \beta$.

E2. Prove that the following are equivalent for any consistent first order theory T in a finite $\mathcal{L}$ which is not finitely axiomatizable:

a. T has a computable set of independent axioms.
b. T has a set of axioms which can be computably enumerated as $\theta_0, \theta_1, \theta_2, \ldots$, so that for every n, $\theta_{n+1} \to \theta_n$ is logically valid, but $\theta_n \to \theta_{n+1}$ is not.
c. Whenever $\rho_0, \rho_1, \rho_2, \ldots$ is any computable enumeration of some set of axioms for T, there exists a strictly increasing computable function $f : \omega \to \omega$ such that for every n:

$$\bigwedge_{k \leq n} \rho_k \qquad \text{does not imply} \qquad \bigwedge_{k \leq f(n)} \rho_k \quad .$$

In (a): a set of axioms Σ for T is "*independent*" if no $\varphi \in \Sigma$ is provable from $\Sigma \backslash \{\varphi\}$.

E3. A *dense tree* is a model $(T, \leq, \wedge)$ where $\leq$ is a partial ordering and $\wedge$ is a meet operation (i.e., $x \wedge y$ is the greatest lower bound of x and y) with the following additional properties:

(1) For each $x \in T$, the set of predecessors $\{y \mid y < x\}$ is a dense linear order without endpoints.
(2) For each $x \in T$, there are $y, z > x$ with $x = y \wedge z$.
(3) For any pairwise incomparable $x, y, z \in T$, exactly two of $x \wedge y$, $x \wedge z$, and $y \wedge z$ are equal.

Show that there is exactly one countable dense tree (up to isomorphism).

Computability Theory

C1. A function $f : \omega \to \omega$ is an *order function* if it is total, nondecreasing and has unbounded range. Show that the set of all $e \in \omega$ such that φ_e is an order functions is Π^0_2-complete.

C2. Show that no computable set C can have the property that both it and its complement are the sets of fixed points of two computable functions f and g; i.e., it is impossible that $C = \{e \mid \varphi_e = \varphi_{f(e)}\}$ and $\overline{C} = \{e \mid \varphi_e = \varphi_{g(e)}\}$.

C3. An enumeration of a collection $\mathcal{C}$ of subsets of ω is a set A such that $\mathcal{C} = \{A^{[e]} \mid e \in \omega\}$ (where $A^{[e]} = \{x \mid \langle x, e\rangle \in A\}$).

(1) Show that there is no computable enumeration of the collection of all computable sets.

(2) Show that there is a computably enumerable enumeration of the collection of all computable sets *without repetition* (i.e., the set A above satisfies $A^{[e]} \neq A^{[i]}$ for all distinct e and i).

Model Theory

M1. Is the following situation possible? Either give an example, or give a proof that it's not possible:

1. $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$–structures with $\mathfrak{A} \prec \mathfrak{B}$.
2. $R \notin \mathcal{L}$ is a new n–place relation symbol (with $1 \le n < \omega$), and T is an $\mathcal{L} \cup \{R\}$ theory.
3. There is a unique $R_A \subseteq A^n$ such that $(\mathfrak{A}; R_A) \models T$.
4. There is a unique $R_B \subseteq B^n$ such that $(\mathfrak{B}; R_B) \models T$.
5. $R_B \cap A^n \neq R_A$.

M2. Let Σ be the theory of infinite dimensional vector spaces over algebraically closed fields of characteristic 0. To formalize Σ, let $\mathcal{L} = \{S, V, +, \cdot\}$, where S, V are unary predicates and $+, \cdot$ are binary functions. Then Σ says that S, V partition the universe into two disjoint non-empty sets, and that the scalars, S, form an algebraically closed field of characteristic 0, and the vectors, V form an infinite dimensional vector space over S. Models will then have both a zero scalar, $0 \in S$, and a zero vector, $\vec{0} \in V$. Define $x + y$ and $x \cdot y$ to be 0 when it would normally be nonsense; for example, $x + y = 0$ when $x \in S$ and $y \in V$, and $x \cdot y = 0$ when $x, y \in V$. Then:

1. Prove that Σ is complete and ω–stable.
2. Prove that Σ is not κ–categorical for any $\kappa \ge \aleph_0$.
3. Prove that Σ has $\le \kappa$ models of size κ for all $\kappa \ge \aleph_0$.
4. For which $\kappa \ge \aleph_0$ does Σ have exactly κ models of size κ?

M3. Assume that $\mathcal{L}$ is countable and contains the symbol $<$. Let Σ be a complete $\mathcal{L}$–theory such that for some $\mathfrak{A} \models \Sigma$, $<_{\mathfrak{A}}$ well-orders A in type ω_1. For any non-well-ordered $\mathfrak{B} \models \Sigma$, let the ordinal $W(\mathfrak{B})$ be its well-ordered initial segment; so $<_{\mathfrak{B}}$ consists of a well-order of type $W(\mathfrak{B})$, followed by a total order with no least element. Then prove:

1. For any non-well-ordered $\mathfrak{B} \models \Sigma$, $W(\mathfrak{B})$ is a limit ordinal.
2. The set of all $W(\mathfrak{B})$ such that $\mathfrak{B}$ is a countable non-well-ordered model of Σ is unbounded in ω_1.

S1. Assume MA($\aleph_1$), and let E be an uncountable subset of the plane, $\mathbb{R}^2$. Prove that there is a Cantor set $K \subseteq \mathbb{R}^2$ such that K contains uncountably many points of E.

K is a Cantor set iff K is homeomorphic to the standard middle-third Cantor set (or to 2^ω).

S2. For this problem, a *nice tree* is a set T of non-empty closed subsets of $[0,1]$ such that for all $H, K \in T$: $H \subseteq K$ or $K \subseteq H$ or $K \cap H = \emptyset$. Note that then for each $K \in T$: $\{H \in T : H \supset K\}$ is totally ordered by $\supset$ (although not necessarily well-ordered). Prove that there is a nice tree which is an Aronszajn tree under the order $\supset$ (that is, $\supsetneq$).

Remarks. So, now each $\{H \in T : H \supset K\}$ will be well-ordered by $\supset$ in some countable type. The root node, which is the largest set in T, could be $[0,1]$, although it might be simpler to make all the nodes Cantor sets. Level α of T is $\mathcal{L}_\alpha := \{K \in T : \text{type}(\{H \in T : H \supset K\}) = \alpha\}$. Then $\mathcal{L}_\alpha$ must be countable (by the definition of "Aronszajn"). It is natural to construct $\mathcal{L}_\alpha$ inductively, but you have to be careful that the construction doesn't die at limit ordinals.

S3. Use the following (standard) definition of the constructible sets:

$$\begin{aligned} L(0) &= \emptyset \\ L(\alpha+1) &= \mathcal{D}(L(\alpha)) \\ L(\gamma) &= \bigcup_{\alpha<\gamma} L(\alpha) \text{ for limit } \alpha \end{aligned}$$

Here, $\mathcal{D}(A)$ is the set of all subsets of A which are (first-order) definable in $(A; \in)$ using finitely many elements of A as parameters. Let $\mathcal{D}^-(A) \subseteq \mathcal{D}(A)$ be the set of those subsets of A which are definable in $(A; \in)$ *without* using any parameters; so $\mathcal{D}^-(A)$ is always countable. Prove that the set of all α such that $L(\alpha+1) = \mathcal{D}^-(L(\alpha))$ is an unbounded subset of $(\omega_1)^L$.

Answers

E1. For (b), let $\beta = \omega + \omega + 2$ and $\alpha = \omega + 1$.

For (a): Find an ordinal δ and positive integer n such that

$$\omega^\delta \cdot n \leq \beta < \omega^\delta \cdot (n+1).$$

Ordinals of the form ω^δ are strongly indecomposable, i.e, if $X \cup Y = \omega^\delta$, then at least one of X or Y has order type ω^δ. Write

$$\omega^\delta \cdot n = L_1 + L_2 + \cdots L_n$$

where each L_i has order type ω^δ. Then for each i either $X \cap L_i$ or $Y \cap L_i$ has order type ω^δ. So there exists $\Sigma \subseteq \{1, \ldots, n\}$ with $|\Sigma| \geq n/2$ such that either

(1) $X \cap L_i$ has order type ω^δ for all $i \in \Sigma$ or
(2) $Y \cap L_i$ has order type ω^δ for all $i \in \Sigma$.

Then since $3|\Sigma| \geq n+1$ we have the result.

E2. $(a) \to (b)$: Given an independent set of axioms $\{\varphi_n : n < \omega\}$ take

$$\theta_n = \bigwedge_{k \leq n} \varphi_k.$$

$(b) \to (c)$: Recall that there is an effective enumeration of all logical validities. For any n we can effectively find some θ_m such that

$$\theta_m \to \bigwedge_{k \leq n} \rho_k$$

is a validity. Since the ρ's are axioms for T we can effectively find some $f(n)$ such that

$$\bigwedge_{k \leq f(n)} \rho_k \to \theta_{m+1}.$$

$(c) \to (b)$: Take a sequence $l_{n+1} = f(l_n)$ and put $\theta_n = \bigwedge_{k \leq l_n} \rho_k$.

$(b) \to (a)$: Put $\varphi_0 = \theta_0$ and $\varphi_{n+1} = (\theta_n \to \theta_{n+1})$.

E3. A representation of a countable dense tree is the set of all functions of the form $f : \mathbb{Q} \cap (-\infty, r) \to \{0, 1\}$ which take the value 1 for only finitely many arguments and where $r \in \mathbb{Q}$, ordered by extension.

Uniqueness follows by a Cantor-style back-and-forth argument: Suppose you have a finite partial isomorphism p from a countable dense tree T_1 to a countable dense tree T_2. Fix $a \in T_1 - \mathrm{dom}(p)$. (The argument for $b \in T_2 - \mathrm{ran}(p)$ is symmetric.) Then there are two cases:

Case 1: There is some (least) $t \in \mathrm{dom}(p)$ with $a < t$: Then the collection $U = \{t' \wedge t \mid t' \in \mathrm{dom}(p) \text{ and } t' \not\geq t\}$ is linearly ordered, and we can use the density of T_2 and clause (3) to find an image for a among or between the images of $p(U)$.

Case 2: Otherwise and there is some (greatest) $t \in \mathrm{dom}(p)$ with $a > t$: Then use clause (2) to find an image for a.

Case 3: Otherwise: Then a is incomparable with all of $\mathrm{dom}(p)$, so fix an element below the meet of $\mathrm{ran}(p)$ and use clause (2) to find an image for a.

C1. Let I be the set of all e such that φ_e is an order function. Then I is clearly Π^0_2. Now, let $A \subseteq \omega$ be any Π^0_2 set; we show that $A \leq_m I$. Write A as $\{x : \forall y \exists z P(x, y, z)\}$, where P is computable. Then there is a computable $\Gamma : \omega \to \omega$ such that for all x, $\varphi_{\Gamma(x)}$ is the (partial) function

$$t \mapsto \sum_{y \leq t} [1 + \mu z P(x, y, z)] \quad .$$

If $x \in A$, then $\varphi_{\Gamma(x)}$ is total and $\Gamma(x) \in I$. If $x \notin A$, then $\Gamma(x) \notin I$ because $\varphi_{\Gamma(x)}$ is not total (and has finite domain).

C2. Given C, f, g, let $h(e, x)$ be the partial computable function such that if $e \in C$, then $h(e, x) = \varphi_{g(e)}(x)$ for all x; and if $e \notin C$, then $h(e, x) = \varphi_{f(e)}(x)$ for all x. By the Recursion Theorem, fix e such that φ_e is the (partial) function $x \mapsto h(e, x)$. If $e \in C$, then $\varphi_e = \varphi_{g(e)}$, but then $e \in \overline{C}$, a contradiction. If $e \in \overline{C}$, then $\varphi_e = \varphi_{f(e)}$, but then $e \in C$, also a contradiction.

C3.

(1) By simple diagonalization.
(2) See Odifreddi I for the proof of the Friedberg theorem of an enumeration of all c.e. sets without repetition and its variations.

M1. This question is due to J.Millar.

There are many examples here, all revolving around the observation that a relation implicitly definable in a given model need not be explicitly definable. Note that Beth's Theorem requires that the relation be implicitly defined in all models of a theory.

Specifically, let $\mathcal{L} = \{<\}$, and let R be unary. Let T_0 be the theory of dense total orders without first or last elements. Then T_0, as an $\mathcal{L}$–theory, is complete and model-complete. Let T be T_0 plus the statement that R is an un-realized Dedekind cut; so T says of a model $(\mathfrak{A}; R_A)$ that R_A and $A \backslash R_A$ are both non-empty, R_A is an initial segment with no largest element and $A \backslash R_A$ is a final segment with no smallest element.

A given dense total order may have no subsets satisfying T (e.g, $\mathbb{R}$) or infinitely many subsets satisfying T (e.g, $\mathbb{Q}$).

Now, let $A = (-\infty, 0) \cup (1, 2) \cup [4, \infty) \subset \mathbb{R}$, with the usual order of real numbers. Then there is a unique $R_A \subseteq A$ such that $(\mathfrak{A}; R_A) \models T$; namely, $R_A = (-\infty, 0)$; note that $(1, 2) \cup [4, \infty) \cong \mathbb{R}$ has no unrealized

Dedekind cuts. Let $B = \mathbb{R}\backslash\{3\}$. Then there is a unique $R_B \subseteq B$ such that $(\mathfrak{B}; R_B) \models T$; namely, $R_B = (-\infty, 3)$. Clearly, $R_B \cap A \neq R_A$.

M2. First note that every $\mathfrak{A} \models \Sigma$ of size $\aleph_0$ or $\aleph_1$ has an elementary extension $\mathfrak{B}$ of size $\aleph_1$ whose S has transcendence degree $\aleph_1$ and whose $\dim(V) = \aleph_1$. Since such a $\mathfrak{B}$ is unique up to isomorphism, Σ must be complete. Also, if Σ failed to be ω–stable, there would be a countable $E \subset B$ such that $\mathfrak{B}$ realizes $\aleph_1$ 1–types over E; but this is impossible because the group of automorphisms of $\mathfrak{B}$ pointwise fixing E has only countably many orbits.

For (2)(3): Note that the $\mathfrak{A} \models \Sigma$ of size $\kappa = \aleph_\alpha$ are characterized by the transcendence degree λ of S, which is some (possibly finite) cardinal $\leq \kappa$, and $\dim(V)$, which is some infinite cardinal $\leq \kappa$; of course, $\max(\lambda, \dim(V), \aleph_0) = \kappa$. There are exactly $\max(\alpha, \aleph_0) \leq \kappa$ such models.

Now (4) follows: $\max(\alpha, \aleph_0) = \kappa$ iff $\kappa = \aleph_0$ or $\kappa = \aleph_\kappa$.

M3. For (1), just note that $\mathfrak{A}$ satisfies the statement that every element has a successor.

For (2): WLOG, $A = \omega_1$, with $<_{\mathfrak{A}}$ the usual order. Let C be the set of $\gamma < \omega_1$ such that the ordinals $< \gamma$ form an elementary submodel of $\mathfrak{A}$; then C is unbounded in ω_1 (and also closed). It is sufficient to show that for each $\gamma \in C$, there is a countable non-well-ordered $\mathfrak{B} \models \Sigma$ with $W(\mathfrak{B}) \geq \gamma$; in fact, one can get $W(\mathfrak{B}) = \gamma$, but that's not required by the problem. Fix $\gamma \in C$, and let $\mathfrak{G} \prec \mathfrak{A}$ be the model built on γ. Since $\mathfrak{G}$ is countable and satisfies the axioms that the ordering of its universe is regular, a standard argument produces an elementary end extension $\mathfrak{H}$ of $\mathfrak{G}$. $\mathfrak{H}$ is usually built using the elementary diagram of $\mathfrak{G}$, together with one new constant c which is larger than all elements of G. The Omitting Types Theorem is used to guarantee that $\mathfrak{H}$ is really an end extension. This $\mathfrak{H}$ could conceivably be well-ordered. To avoid this, modify the standard argument to use new constants c_n for $n \in \omega$, where each c_n is larger than all elements of G and $c_0 > c_1 > c_2 > \cdots$.

S1. Note that you cannot simply quote the fact that E must be of first category, since a closed nowhere dense subset of the plane might be connected.

Proof 1. (specific to the plane): WLOG, $|E| = \aleph_1$. By changing coordinates, we may assume that the coordinate axes are not parallel to any of the $\aleph_1$ lines through pairs of points in E. Then, the coordinate projections π_1 and π_2 are 1-1 on E. In the line, every first category set is covered by countably many Cantor sets, so choose a Cantor set $K_1 \subseteq \mathbb{R}$ such that $E \cap \pi_1^{-1}(K_1)$ is uncountable. Repeat the argument and choose a Cantor set $K_2 \subseteq \mathbb{R}$ such that $(E \cap \pi_1^{-1}(K_1)) \cap \pi_2^{-1}(K_2)$ is uncountable. Let $K = K_1 \times K_2$.

Proof 2. (works in any Polish space X). Force a generic Cantor set K using a finitely branching tree of basic open sets.

S2. As indicated in the Remarks, we construct $\mathcal{L}_\alpha$ inductively, with all nodes in the tree Cantor subsets of $[0,1]$. $\mathcal{L}_0$ can be a singleton (the root node), and each node will have $\aleph_0$ children. Note that the tree will automatically be Aronszajn because there cannot be a decreasing ω_1–sequence of closed sets. The problem is that the construction doesn't die at a limit ordinal.

Say $\gamma < \omega_1$ is a limit and we have $\mathcal{L}_\alpha$ for $\alpha < \gamma$. Let $T_\gamma = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha$, which is a countable tree of height γ. Whenever $\mathcal{C}$ is a maximal chain in T_γ, $\bigcap \mathcal{C}$ is a closed subset of 2^ω, and is non-empty by compactness, although it may be a singleton. If all these $\bigcap \mathcal{C}$ are singletons (or countable), the construction cannot proceed.

To ensure that suitably many of the $\bigcap \mathcal{C}$ are Cantor sets, we inductively maintain the following condition: Whenever $\alpha < \beta < \omega_1$ and $F \in \mathcal{L}_\alpha$ and $\mathcal{P}$ is a finite family of non-empty clopen subsets of F, there is a $G \in \mathcal{L}_\beta$ such that $G \subset F$ and $G \cap P \neq \emptyset$ for all $P \in \mathcal{P}$.

S3. We can work within L; so WLOG $V = L$, and we're proving that $S := \{\alpha : L(\alpha+1) = \mathcal{D}^-(L(\alpha))\}$ is an unbounded subset of ω_1. Now $S \subseteq \omega_1$ since $L(\alpha) \subset L(\alpha+1)$ and $\mathcal{D}^-(L(\alpha))$ must be countable. If S is bounded, let $\beta = \sup\{\alpha + 1 : \alpha \in S\} < \omega_1$. Let $M \prec L(\omega_1) = H(\omega_1)$ be the Skolem hull of $\emptyset$ in $L(\omega_1)$ using the definable Skolem functions. Then M is transitive and of the form $L(\gamma)$. Each element of M is definable in M (without parameters), so $\gamma \in S$ and hence $\gamma < \beta$. But β is definable in $L(\omega_1)$, so $\beta \in L(\gamma)$, a contradiction.