Qualifying Exam
Logic
August 2008

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. (Prove or disprove) A total order is well-ordered iff every suborder of it is isomorphic to an initial segment.

E2. Let A and B be disjoint infinite sets of positive integers. Let (M, R) be the following model. The universe is

$$M = A \cup B \cup (A \times B)$$

and R is the following ternary relation:

$$R = \{(a, b, (a, b)) : a \in A, b \in B\}.$$

Prove that $T = Th(M, R)$ is not finitely axiomatizable.

E3. Let T be a consistent axiomatizable theory with only finitely many complete extensions in the same language. Show that T is decidable. (Here, a theory is a set of sentences closed under deduction, and it is axiomatizable if it is the deductive closure of a computable set of sentences.)

Computability Theory

C1. Prove that if $A \leq_{wtt} B$, then $A \leq_{tt} B \oplus \emptyset'$.

Recall that:

A set A is *truth-table reducible* to a set B, abbreviated *tt-reducible* ($A \leq_{tt} B$), if there are recursive functions f and g such that $x \in A$ if and only if $B \upharpoonright f(x) = D_y$ for some $y \in D_{g(x)}$.

A set A is *weak truth-table reducible* to a set B, abbreviated *wtt-reducible* ($A \leq_{wtt} B$), if there are recursive functions f and g such that $A(x) = i$ if and only if $B \upharpoonright f(x) = D_y$ for some $y \in W_{g(x,i)}$.

C2. Prove that no Δ^0_2-complete set exists.

Hint: Let A be Δ^0_2. Build another Δ^0_2 set $B \not\leq_m A$ by diagonalization.

C3. Show that every hyperimmune set H is a subset of some 1-generic G.

Hint: Build G by finite extensions. Given an approximation $\tau \in 2^{<\omega}$ to G, look at extensions of $\tau 1^n$.

Recall that a set H is hyperimmune iff for any computable strictly increasing sequence $(n_k : k < \omega)$ there exists a k such that the interval $[n_k, n_{k+1})$ is disjoint from H. A set G is 1-generic iff for any computable enumerable set $W \subseteq 2^{<\omega}$ there exists $\tau \subseteq G$ an initial segment such that either $\tau \in W$ or no extension of τ is in W.

M1. Let $\mathcal{L}$ be a first order language without function symbols. For an $\mathcal{L}$-structure $\mathfrak{A}$ realizing a complete 1-type $\Gamma(x)$, define $\mathfrak{A}\restriction\Gamma$ to be the substructure of $\mathfrak{A}$ whose universe is $\{a \in A : a \text{ realizes } \Gamma \text{ in } \mathfrak{A}\}$.

Prove or disprove:

For $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$, if $\mathfrak{A}\restriction\Gamma$ and $\mathfrak{B}\restriction\Gamma$ are both countably infinite and $\aleph_0$-categorical, then $\mathfrak{A}\restriction\Gamma \cong \mathfrak{B}\restriction\Gamma$.

M2. Let $\mathcal{L} = \{<\}$ and let Σ be a complete theory in $\mathcal{L}$. Assume that Σ has infinite models and that Σ includes the axioms that $<$ is a (strict) partial order and is tree-like — that is,

$$\forall x, y, z\, [x < z \wedge y < z \rightarrow [x = y \vee x < y \vee y < x] \;\; .$$

Assume that no model of Σ has an infinite chain. Prove that Σ is κ–stable for all $\kappa \geq 2^{\aleph_0}$.

M3. Let M be an ordered field. Define $x \equiv^M y$ iff x and y satisfy all formulas $|x - y| \leq 1/n$ for $n = 1, 2, \ldots$.

Prove that if M is sufficiently saturated, then the linear order

$$((0 \leq x \leq 1)^M / \equiv^M)$$

is isomorphic to the unit interval of the real line.

S1. Assume MA($\aleph_1$). Assume that $x_\alpha, y_\alpha \in [\mathbb{Q}]^\omega$ for $\alpha \in \omega_1$ and $x_\alpha \perp y_\beta$ for all α, β. Assume further that each x_α and y_β is a convergent ω–sequence in $\mathbb{R}$, with x_α converging to r_α and y_β converging to s_β, where $r_\alpha \neq s_\beta$ and $r_\alpha, s_\beta \notin \mathbb{Q}$ for all α, β. Prove that for some $c \in [\mathbb{Q}]^\omega$: $x_\alpha \subseteq^* c$ and $y_\beta \perp c$ for all α, β.

Notation. $[\mathbb{Q}]^\omega$ is the collection of all infinite sets of rationals. For $x, y \in [\mathbb{Q}]^\omega$, $x \perp y$ means that $x \cap y$ is finite, and $x \subseteq^* y$ means that $x \backslash y$ is finite.

S2. Let M be a countable transitive model of ZFC. Fix $T \in M$ such that $M \models$ "T is an ω_1–Aronszajn tree". Let $\mathbb{P} \in M$ be the forcing poset of finite partial functions from κ to 2 (so $\kappa \in M$). Let G be $\mathbb{P}$–generic over M. Prove that $M[G] \models$ "T is an ω_1–Aronszajn tree".

S3. Assume $V = L$. Let γ be the least ordinal such that $L(\gamma) \equiv L(\omega_1)$. Prove that $L(\gamma) \prec L(\omega_1)$.

Answers

E1. This is true. If the order fails to be a well-order, write it as $\alpha + E$, where $\alpha \in ON$ and $E \neq \emptyset$ is a total order with no least element. But, if $e \in E$, then $\alpha \cup \{e\}$ cannot be isomorphic to an initial segment of $\alpha + E$.

E2. Since a natural number is never an ordered pair, the three sets $A, B, (A \times B)$ are pairwise disjoint. Thus, we may axiomatize T by saying that:

1. The universe is partitioned into three disjoint sets,
$A := \{x : \exists y, z\, R(x, y, z)\}$, $B := \{y : \exists x, z\, R(x, y, z)\}$,
$C := \{z : \exists x, y\, R(x, y, z)\}$.
2. R defines a bijection between $A \times B$ and C.
3. For each $n \in \omega$: A and B each have at least n elements.

These axioms are complete, since they are $\aleph_0$–categorical; hence, they do indeed axiomatize T. Since axioms (1)(2) alone have models in which $|A| = |B| = n$ and $|C| = n^2$, no finite subset of (1)(2)(3) can axiomatize T, so T is not finitely axiomatizable.

E3. Since any two complete extensions can be distinguished by a sentence there must be $n < \omega$ and sentences θ_i for $i < n$ such that $T \cup \{\theta_i\}$ for $i < n$ is a list of all complete extensions of T. Complete axiomatizable theories are decidable. To decide if $T \vdash \theta$ simple check that $T \cup \{\theta_i\} \vdash \theta$ for all $i < n$.

C1. Let f and g be the computable functions witnessing $A \leq_{wtt} B$. Note there there is a computable function h such that $y \in W_{g(x,1)}$ iff $h(x, y) \in \emptyset'$. Define

$$\hat{f}(x) = 2 \max\{f(x), h(x, y) \colon D_y \subseteq \{0, \ldots, f(x) - 1\}\} + 1.$$

This is defined so that we can determine the value of $A(x)$ from $B \oplus \emptyset' \restriction \hat{f}(x)$. In particular, define $D_{\hat{g}(x)}$ to be the set of all z such that there is a y for which:

- $D_z \subseteq \{0, \ldots, \hat{f}(x) - 1\}$,
- $D_y = \{n < f(x) \colon 2n \in D_z\}$, and

- $2h(x,y)+1 \in D_z$.

Then $\hat{f}$ and $\hat{g}$ witness $A \leq_{tt} B \oplus \emptyset'$.

C2. Let

$$B = \{e : \varphi_e(e)\downarrow \text{ and } \varphi_e(e) \notin A\}.$$

Then B is Turing reducible to $A \oplus \emptyset'$ hence it is Δ^0_2. But it is not many-one reducible to A.

C3. (Jockusch) Suppose $W \subseteq 2^{<\omega}$ is computable enumerable and τ is an approximation to G.

Case 1. There exists n such that $\sigma \notin W$ for all $\sigma \supseteq \tau 1^n$. We take any such $\tau 1^n$ to be our next approximation to G.

Case 2. Not case 1. We build a computable strictly increasing sequence n_k so for each k there exists a σ_k in W of length less than n_{k+1} which extends $\tau 1^{n_k}$. By hyperimmunity there exists k such that the interval $[n_k, n_{k+1})$ is disjoint from H. We take any such σ_k as the next approximation to G.

M1. This is false. For example, let $\mathcal{L} = \{<\} \cup \{P_n : n \in \omega\}$, where each P_n is unary. Let Σ be the theory which says that $<$ is a dense total order without endpoints, each P_n is a non-empty proper initial segment without a largest element, each $P_{n+1} \subsetneqq P_n$, and the complement of each P_n has no smallest element. Then Σ is complete and $\Sigma \cup \{P_n(x) : n \in \omega\}$ defines a (complete) 1-type $\Gamma(x)$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be built on the rationals, but let $\mathfrak{A}\restriction\Gamma$ be a non-empty initial segment *with* a largest element, and let $\mathfrak{B}\restriction\Gamma$ be a non-empty initial segment *without* a largest element.

M2. If T has no model with an infinite chain, then by compactness there must be $n < \omega$ such that no model of T contains a chain of length n. We show that T is κ-stable for any κ of size at least the continuum. Suppose for contradiction that T has a model $\mathfrak{A}$ with contains a set X of size κ and Y of size κ^+ such that the type over X of each element of Y is distinct. Say that two elements of $\mathfrak{A}$ are in the same component iff there are above the same minimal element. Then by cutting down Y we may assume that either

Case 1. All elements of Y are in the same component.

or

Case 2. Elements of Y are in distinct components.

Assume case 1. Since components of $\mathfrak{A}$ are trees of height less than n we may assume (by cutting down Y) that there exists a node w such that for all

$u \in Y$ there exists an immediate child c_u of w such that $w < c_u \leq u$ and for distinct $u, v \in Y$ we have that $c_u \neq c_v$. For $u \in Y$ let $\mathfrak{A}_u$ be the substructure $\mathfrak{A}$ consisting of $\{v : c_u \leq v\}$. By cutting down Y we may assume the no element of X is in $\mathfrak{A}_u$ for any $u \in Y$. Since κ is at least the continuum we may find distinct $u, v \in Y$ such that $(\mathfrak{A}_u, u)$ is elementary equivalent to $(\mathfrak{A}_v, v)$. But now the obvious Ehrenfeucht game strategy (play the identity outside $\mathfrak{A}_u$ and $\mathfrak{A}_v$), shows that u and v have the same type over X.

A similar argument works for Case 2.

M3. All elements x in $[0,1]^M$ are equivalent to some standard real:

For each $n = 1, 2, \ldots$ there exists $k_n < n$ with

$$\frac{k_n}{n} \leq x \leq \frac{k_n + 1}{n}.$$

Then $x \equiv^M r$ where $r = \sup_n \frac{k_n}{n}$.

Sufficiently saturated implies that this "standard part map" is surjective.

S1. Let

$$\mathbb{P} = \{p \in \mathrm{Fn}(\omega_1, [\mathbb{Q}]^{<\omega}) : \forall \alpha, \beta \in \mathrm{dom}(p)\, [x_\alpha \backslash p(\alpha) \cap y_\beta \backslash p(\beta) = \emptyset]\} \quad .$$

Assuming that $\mathbb{P}$ is ccc, let G be a filter meeting the dense sets $\{p : \alpha \in \mathrm{dom}(p)\}$ for each $\alpha < \omega_1$, and let $F = \bigcup G$. Then each $x_\alpha \backslash F(\alpha) \cap y_\beta \backslash F(\beta) = \emptyset$, so we can let $c = \bigcup\{x_\alpha \backslash F(\alpha) : \alpha < \omega_1\}$.

To prove that $\mathbb{P}$ is ccc, we show that in fact $\mathbb{P}$ is σ–centered. For each $p \in \mathbb{P}$, let $H_p = \bigcup\{x_\alpha \backslash p(\alpha) \cup \{r_\alpha\} : \alpha \in \mathrm{dom}(p)\}$ and let $K_p = \bigcup\{y_\alpha \backslash p(\alpha) \cup \{s_\alpha\} : \alpha \in \mathrm{dom}(p)\}$. Then H_p, K_p are disjoint compact subsets of $\mathbb{R}$, so there are U, V which are finite unions of rational open intervals such that $H_p \subset U$ and $K_p \subset V$. There are only $\aleph_0$ such U, V, and for each such U, V, $\{p : H_p \subset U \ \&\ K_p \subset V\}$ is centered.

S2. Assume that T fails to be an Aronszajn tree in $M[G]$. Since $\mathbb{P}$ is ccc, the failure must be because in $M[G]$, T has an uncountable chain.

In M, there is a name $\mathring{C}$ such that some $p \in \mathbb{P}$ forces $\mathring{C}$ to be an uncountable maximal chain in T. Maximality implies that $\mathring{C}$ meets every level, so for each α, choose $q_\alpha \leq p$ and t_α in level α of T such that $q_\alpha \Vdash t_\alpha \in \mathring{C}$. Now, the proof that this $\mathbb{P}$ is ccc actually yields an uncountable $E \subseteq \omega_1$ such that the p_α, for $\alpha \in E$, are pairwise compatible. So, in M, the set $\{t_\alpha : \alpha \in E\}$ is an uncountable chain, contradicting the assumption that T is Aronszajn.

S3. Let D be the set of elements of $L(\gamma)$ which are definable in $L(\gamma)$ without parameters. Then $D \preceq L(\gamma)$ because $L(\gamma)$ has a definable well-order. If $L(\delta)$ is the transitive collapse of D, then $\delta \leq \gamma$ and $L(\delta) \equiv L(\omega_1)$, so that $\delta = \gamma$. Thus, $D = L(\gamma)$.

Now, for $a \in L(\gamma)$, choose a formula $\varphi_a(x)$ which defines a in $L(\gamma)$, and prove that φ_a also defines a in $L(\omega_1)$. For $a \subseteq \omega$ or $a \subseteq \omega \times \omega$, this is easy by absoluteness of natural numbers and $L(\gamma) \equiv L(\omega_1)$. For general a, use the fact that $L(\gamma) \models$ "all sets are countable", and the fact that a is determined by a relation on ω isomorphic to $\mathrm{trcl}(\{a\})$.

Finally, $L(\gamma) \prec L(\omega_1)$ because $L(\gamma) \models \psi[a_1, \ldots, a_n]$ is equivalent to $L(\gamma) \models \exists x_1, \ldots x_n \, [\psi[x_1, \ldots, x_n] \wedge \varphi_{a_1}(x_1) \wedge \cdots \wedge \varphi_{a_n}(x_n)]$.