Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. For an abelian group G and prime p we say that G is divisible by p iff for every $x \in G$ there is a $y \in G$ such that $py = x$. Prove that there is a computable $G \subseteq \mathbb{Q}$ which is a subgroup of $(\mathbb{Q}, +)$ but

$$\{p \ : \ G \text{ is divisible by } p\}$$

is not computable.

E2. For each prove or disprove:

(a) There exists a set D of reals with the same order type as the rationals which is a closed subset of the real number line.

(b) There exists a set D of reals with the same order type as the rationals which is discrete, i.e., no point of D is a limit point of D.

(c) There exists a set D of reals with the same order type as the rationals such that every point of D is a limit point of D but only from below and not above.

E3. Let $B = \bigcup_{i \in \omega} [p_i, q_i] \subset \mathbb{R}$, where each $p_i < q_i$, the intervals $[p_i, q_i]$ are pairwise disjoint, and all p_i, q_i are rational. Let $A = B \cap \mathbb{Q}$. View A, B as structures for $\mathcal{L} = \{<\}$. Prove that A is an elementary substructure of B.

Computability Theory

C1. Show that the intersection of two hyper-simple sets is hyper-simple.

Recall that A is *hyper-simple*[1] iff it is c.e., coinfinite and there is no computable function f such that for all n, $f(n)$ is greater than the nth element of the complement of A, i.e., the complement is hyper-immune.

C2. Let S be a class of c.e. sets closed under finite variation that contains the computable sets but not all the c.e. sets. Let $I = \{e : W_e \in S\}$ and let A be any Π^0_2-set. Prove that $A \leq_m I$.

Comment: In fact, this can be shown for any Σ^0_3-set A as well.

C3. Show that every non-computable c.e. set computes a 1-generic.

Recall that a set $G \in 2^\omega$ is 1-*generic* iff for any computably enumerable set $D \subseteq 2^{<\omega}$ there exists τ an initial segment of G such that either $\tau \in D$ or no extension of τ is in D.

[1] Of course, if you prefer, you may use the definition of hyper-simple in terms of disjoint strong arrays, as it is given in Soare's book.

Set Theory

S1. Let λ be an infinite cardinal, and assume MA(λ). Let $\mathbb{P}$ be a ccc poset. Let $D_\alpha \subseteq \mathbb{P}$ be dense in $\mathbb{P}$ for $\alpha < \lambda$, and fix $E \subseteq \mathbb{P}$ with $|E| \leq \lambda$. Prove that there are filters $G_n \subseteq \mathbb{P}$ for $n < \omega$ such that $G_n \cap D_\alpha \neq \emptyset$ for each n and α, *and* such that $E \subseteq \bigcup_n G_n$.

Note that without the E, this is obvious by the definition of MA(λ).

S2. Let T be any special Aronszajn tree, and let L_α be the α^{th} level of T for $\alpha < \omega_1$. Prove that there is a map $\varphi : T \to \mathbb{Q}$ such that φ is order-preserving ($x < y \to \varphi(x) < \varphi(y)$) and such that $\varphi \restriction L_\alpha$ is a 1-1 function for each α.

A *tree* is a partial order $(T, \leq)$ such that $\{y \in T \ : \ y < x\}$ is well-ordered by $<$ for each $x \in T$. The *level* of x is the corresponding ordinal. A tree T is *Aronszajn* iff T does not have an uncountable chain but L_α is nonempty and countable for every $\alpha < \omega_1$. An Aronszajn tree is *special* iff it is a countable union of antichains.

S3. In the ground model, let $\mathbb{P}$ be a ccc poset, and assume that $\mathbb{1}$ forces that $\mathbb{P}$ adds a new subset of ω_2. Prove that $\mathbb{1}$ forces that $\mathbb{P}$ adds a new subset of ω_1.

Answers

E1. Take a set P of primes which is computably enumerable but not computable with computable enumeration P_s and look at the group generated by the set of $1/p^s$ where p is a prime not in P_s.

E2. (a) This is impossible. $\mathbb{Q}$ is not Dedekind complete, so fix $A \subset D$ so that A is bounded above but has no least upper bound in D. Then in $\mathbb{R}$, $\sup(A) \notin D$, so D is not closed.

(b) Copy the proof that every countable order type embeds into $\mathbb{R}$. List $\mathbb{Q}$ as $\{q_n : n \in \omega\}$ and choose disjoint closed intervals $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $q_n < q_m$ iff $I_n < I_m$. Then let $D = \{(a_n + b_n)/2 : n \in \omega\}$.

(c) Choose the I_n as in (b), but make sure that $\bigcup_n I_n$ is dense in $\mathbb{R}$ (for example, make sure that $\mathbb{Q} \subset \bigcup_n I_n$). Then let $D = \{a_n : n \in \omega\}$.

E3. Observe that for all $a_1, \ldots, a_n \in A$ and $b \in B$, there is an automorphism α of B such that $\alpha(b) \in A$ and each $\alpha(a_i) = a_i$, since you can always move an irrational to a nearby rational by a piecewise linear map. Now, apply the Tarski–Vaught criterion for elementary submodel.

C1. Assume A and B are c.e. and coinfinite but $A \cap B$ is not hyper-simple. So there is a computable function f such that, for all n, the nth element of the complement of $A \cap B$ is less than $f(n)$. Consider the c.e. set S of all n such that the nth element of the complement of A is *not* less than $f(2n-1)$. If S is finite, then a finite variation of $f(2n-1)$ witnesses that A is not hyper-simple. So assume that S is infinite. For $n \in S$, there are at least n elements in the complement of B less than $f(2n-1)$. Define a computable function g as follows. To compute $g(m)$, find $n \in S$ with $n \geq m$ and let $g(m) = f(2n-1)$. Then g witnesses that B is not hyper-simple.

C2. Let W be a c.e. set not in S and let $A = \{m \colon (\forall n)(\exists k)\ R(m,n,k)\}$, where R is computable. Let f be a computable map such that

$$W_{f(m)} = W \cup \{q \colon (\forall n \leq q)(\exists k)\ R(m,n,k)\}.$$

If $m \in A$, then $W_{f(m)} = \omega$. If $n \notin A$, then $W_{f(m)} =^* W$. Thus $A \leq_m I$.

C3. Let A_s be a uniformly computable enumeration of A. Construct sequence $(\tau_n : n < \omega)$ of elements of $2^{<\omega}$ computable in A as follows. At stage n let s be the least such that $A_s \cap n = A \cap n$. Look for an $e < n$ such that $W_{e,s} \cap \{\rho : \rho \subseteq \tau_n\}$ is empty but there exists $\rho \in W_{e,s}$ such that $\tau_n \subseteq \rho$. For the least such e (if there is one) put $\tau_{n+1} = \rho$ for the least such ρ.

S1. If $\lambda = \aleph_0$, this is easy by the Generic Filter Existence Theorem, so assume that $\lambda > \aleph_0$. Let $\mathbb{Q}$ be the set of all $\vec{q} \in \mathbb{P}^\omega$ such that $q_n = \mathbb{1}$ for all but finitely many n. Order $\mathbb{Q}$ coordinate-wise; then $\mathbb{Q}$ is ccc by MA($\aleph_1$). Let H be a filter on $\mathbb{Q}$ meeting the dense sets $\{\vec{q} : q_n \in D_\alpha\}$ for each n, α, and also meeting $\{\vec{q} : \exists n\, [q_n \leq e]\}$ for each $e \in E$.

S2. Let $T = \bigcup_{n \in \omega} A_n$, where each A_n is an antichain. WLOG, the A_n are disjoint. Also, since each L_α is countable, WLOG each $|A_n \cap L_\alpha| \leq 1$. Now, define $\varphi \restriction A_n$ in the standard way by induction on n; let $\varphi(A_0) = \{0\}$, $\varphi(A_1) = \{-1, 1\}$, $\varphi(A_2) = \{-3/2,\ -1/2,\ 1/2,\ 3/2\}$, etc.

S3. If this fails, then we have a $p \in \mathbb{P}$ and a name $\mathring{f}$ such that p forces $\mathring{f}$ to be a new function from ω_2 into 2. Working always in the ground model, define a subtree $T \subset 2^{<\omega_2}$, where the nodes at level α of T are those $s : \alpha \to 2$ such that some $q \leq p$ forces $\mathring{f} \restriction \alpha = s$. Then T is a tree of height ω_2 such that all levels of T are countable and non-empty, but T has no cofinal path, which is a contradiction.