Instructions:

Do two E problems and two problems in the area C or M in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let L be a language which includes a unary relation symbol R. Let ϕ be an L-sentence and Γ a set of L-sentences neither of which contains the symbol R. If Γ proves ϕ in the language L, must there be a deduction of ϕ from Γ in which R does not occur (i.e., in the language $L - \{R\}$)? If so, prove that there is always such a deduction; and if not, describe Γ and ϕ which provide a counterexample.

E2. Show that there exists an $\mathcal{N} \models \mathrm{PA}$ and an $a \in \mathcal{N} \setminus \mathbb{N}$ so that a is definable in $\mathcal{N}$.

E3. Let α, β and γ be ordinals. Prove that the six sums,

$$\begin{array}{ll} \alpha+\beta+\gamma, & \alpha+\gamma+\beta, \\ \beta+\alpha+\gamma, & \beta+\gamma+\alpha, \\ \gamma+\alpha+\beta, & \gamma+\beta+\alpha, \end{array}$$

cannot all be different.

Computability Theory

C1. Say that a computable function f *has a limit* if for all x, $\lim_s f(x,s)$ exists. Show that the index set $\{e \mid \phi_e \text{ has a limit}\}$ is Π_3-complete.

C2. Show that no 1-generic set computes a diagonally noncomputable function. (Recall that a function f is diagonally noncomputable if for all e, $f(e) \neq \phi_e(e)$.)

C3. Show that $\mathbf{a}$ is a hyperimmune degree if and only if $\mathbf{a}$ computes a function f that agrees with every total computable function infinitely often. (Recall that a Turing degree $\mathbf{a}$ is *hyperimmune* if it computes a function g that is not dominated by any total computable function.)

Model Theory

M1. Let T be a theory in the language of a single unary function f stating that f has no loops (i.e., for every $n > 0$ and every x, $f^n(x) \neq x$) and for every x, there are infinitely many y with $f(y) = x$. Show that T has quantifier elimination, is complete and not κ-categorical for any infinite cardinal κ.

M2. Find a complete theory T in a countable first-order language such that the space $S_1(T)$ of 1-types is uncountable but T is atomic. (Recall that T is *atomic* if every formula $\phi(x_1, \ldots, x_n)$ is contained in a principal n-type.)

M3. Show that a complete countable first-order theory with infinite models is $\aleph_0$-categorical if and only if all of its models are pairwise back-and-forth equivalent.

Recall A and B are back-and-forth equivalent if there is a set I comprised of pairs $(\bar{a}, \bar{b})$ where $\bar{a} \subset A$ and $\bar{b} \subset B$ such that the following hold:

- $(\emptyset, \emptyset) \in I$,
- If $(\bar{a}, \bar{b}) \in I$, then $|\bar{a}| = |\bar{b}| < \omega$ and $\mathrm{tp}^A_{q.f.}(\bar{a}) = \mathrm{tp}^B_{q.f.}(\bar{b})$ (i.e., their quantifier-free types coincide),
- If $(\bar{a}, \bar{b}) \in I$ and $c \in A$, then there exists a $d \in B$ so that $(\bar{a}c, \bar{b}d) \in I$, and
- If $(\bar{a}, \bar{b}) \in I$ and $d \in B$, then there exists a $c \in A$ so that $(\bar{a}c, \bar{b}d) \in I$.

Sketchy Answers or Hints

E1 ans. Straightforward application of the Completeness theorem: If Γ proves ϕ, then any model M of Γ is a model of ϕ. The same then also holds for any model M of Γ in the language $L - \{R\}$, so again by Completeness, there is a deduction of ϕ from Γ in the language $L - \{R\}$.

E2 ans. By the Incompleteness Theorem, we can find a Δ_0-sentence $\phi(x)$ such that $\mathbb{N} \models \forall x\, \neg\phi(x)$ but $\mathrm{PA} + \exists x\, \phi(x)$ is consistent. Then any model $\mathcal{N} \models \mathrm{PA} + \exists x\, \phi(x)$ contains, by induction, a least witness a for ϕ, which must be both nonstandard and definable.

E3 ans. Write α, β and γ in Cantor normal form as

$$\omega^{\alpha_n} \cdot a_n + \cdots + \omega^{\alpha_0} \cdot a_0,\ \omega^{\alpha_n} \cdot b_n + \cdots + \omega^{\alpha_0} \cdot b_0,\ \omega^{\alpha_n} \cdot c_n + \cdots + \omega^{\alpha_0} \cdot c_0,$$

respectively, where $a_n, \ldots, a_0, b_n, \ldots, b_0, c_n, \ldots, c_0$ are non-negative integers. Now use the fact that for $\delta < \epsilon$, $\omega^\delta \cdot d + \omega^\epsilon = \omega^\epsilon$.

C1 ans. It is easy to see that it is Π^0_3. Let $R(n, x, m, t)$ be a total computable predicate. Let $f_n(x, s) =$ the least m such that $(\forall t \leq s)\ R(n, x, m, t)$, or s, if no such m exists. Then f_n has a limit iff $(\forall x)(\exists m)(\forall t)\ R(n, x, m, t)$.

C2 ans. Let G be 1-generic. Let Γ be a Turing functional. We want to prove that Γ^G is not a DNC function. If Γ^G is partial, then there is nothing to show, so assume that it is total. Consider the Σ^0_1 set of strings

$$W = \{\sigma \in 2^{<\omega} \colon (\exists e, s)\ \Gamma^\sigma_s(e) = \phi_{e,s}(e) \text{ (and both converge)}\}.$$

If there is a $\tau \prec X$ such that $\tau \in W$, then Γ^G is not DNC. The only case that remains (thanks to the 1-genericity of G) is that there is a $\tau \prec X$ that has no extension in W. We will show that this is impossible. Define a computable function $f \colon \omega \to \omega$ as follows. To find $f(e)$, search for a $\sigma \succeq \tau$ and an $s \in \omega$ such that $\Gamma^\sigma_s(e) \downarrow$ and let $f(e) = \Gamma^\sigma(e)$. The totality of Γ^G implies that some extension of τ makes Γ converge, so f is total. The fact that τ has no extension in W implies that f is DNC. But no computable function can be DNC, so we have the necessary contradiction.

C3 ans. A function f that agrees with every total computable function infinitely often cannot be dominated by a total computable function. For the other direction, let g be an **a**-computable function that is not dominated by any computable function. The **a**-computable function f defined by

$$f(\langle e,n\rangle) = \begin{cases} \phi_e(\langle e,n\rangle) & \text{if } \phi_{e,g(n)}(\langle e,n\rangle)\downarrow, \\ 0 & \text{otherwise} \end{cases}$$

agrees with every total computable function infinitely often. If it fails on the total computable function ϕ_e, then

$$n \mapsto \text{least } s \text{ such that } \phi_{e,s}(\langle e,n\rangle)\downarrow$$

dominates g.

M1 ans. Proof of QE 1: Let's consider a formula of the form $\exists y(\phi(\bar{x},y))$ where ϕ is a conjunction of literals: $\bigwedge \pm t_1(\bar{x},y) = t_2(\bar{x},y)$. Each term can take in only one parameter (as f is unary), so this really is $\bigwedge \pm t_1(x_i) = t_2(y)$. Whether or not this configuration can hold is determined only by the configuration of $\bar{x}$ - this can be verified in cases: The only hard-ish case is when two x's are connected and $f(x_1) = y$ and $f(y) = x_2$, but $f^2(x_1) \neq x_2$ Proof of QE 2 (the better one): We show that every type $p \in S_1(A)$ is determined by its q.f.-type. Suppose we had a model M with 2 element realizing the q.f.-type p. If the q.f.-type says it's connected to an $a \in A$, then show that the two elements are automorphic in M over A. If it's not connected, then in a saturated elementary extension (which must be homogeneously splitting), it's easy to automorph the two elements while fixing A. Completeness follows from QE-ness. Not $\aleph_0$-categorical: one model with 1 tree and one model with 2 trees. Not $\aleph_1$-categorical: One model with $\aleph_1$-splittings on a single tree, and one model with $\aleph_0$-splittings but $\aleph_1$-many trees.

M2 ans. Take a tree in $2^{<\omega}$ with infinitely many paths but a dense set of isolated paths. Let T be the theory associated to this tree (ie. the 1-types in T are exactly the paths through this tree, and all 2-types are controlled by 1-types. This works

M3 ans. $\leftarrow$: Take any 2 countable models. The back-and-forth builds an isomorphism. $\rightarrow$: Using Ryll-Nardzewski, build the back-and-forth. Given $(\bar{a}, \bar{b}) \in I$ by stage s, and any element $c \in A$, let ϕ isolate the type of c over $\bar{a}$. $\exists x \phi(x)$ is in the type of $\bar{a}$, thus also of $\bar{b}$. Let d be a realization of this formula, and put $(\bar{a}c, \bar{b}d)$ into I at stage $s+1$. Do the back direction too.