

**Instructions:**

Do two E problems and two problems in the area C in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

**E1.** A countable graph  $\mathcal{G} = \langle V, E \rangle$  is *the random graph* if for every pair of disjoint finite sets  $A, B \subseteq V$ , there is a vertex  $v \in V$  such that  $\mathcal{G}$  has edges between  $v$  and every element of  $A$ , but no edge between  $v$  and any element of  $B$ . Note that this determines a unique countable graph. Prove that the theory of the random graph is not finitely axiomatizable.

**E2.** Let  $\delta$  be any ordinal and let  $\gamma = \omega^\delta$  (under ordinal exponentiation). Let  $\mathcal{U} = \{S \subseteq \gamma : S \text{ has order type } \omega \text{ and is unbounded in } \gamma\}$ . Prove that  $|\mathcal{U}|$  is 0 or  $|\gamma|^{\aleph_0}$ .

**E3.** Let  $T$  be a consistent c.e. axiomatizable extension of Peano Arithmetic. Show that there is an  $e$  such that the partial computable function  $\varphi_e$  is total, but  $T$  does not prove that  $\varphi_e$  is total.

## Computability Theory

**C1.** Given  $A \subseteq \omega$ , let  $A^-$  be the set of differences of elements of  $A$ , i.e.,  $A^- = \{|a - b| : a, b \in A\}$ . Characterize the possible Turing degrees of  $A^-$  when  $A$  is a computable set.

**C2.** Let  $A$  have hyperimmune-free degree. Let  $T \subseteq 2^{<\omega}$  be a  $\Delta_2^0$  tree such that  $A \in [T]$ . Show that there is a computable tree  $Q \subseteq 2^{<\omega}$  such that  $A \in [Q] \subseteq [T]$ .

**C3.**

- (a) Show that a computable well-order of  $\omega$  must have a computable strictly ascending sequence.
- (b) Construct an infinite computable linear order with no computable strictly ascending or descending sequence.

## Sketchy Answers or Hints

**E1 ans.** Note that the above definition of the random graph is given by an infinite set  $S$  of axioms. Suppose that there is a finite set  $S_0$  of axioms characterizing the random graph; then there is a finite subset  $S_1 \subset S$  such that  $S_1 \models S_0$ . Now let  $n$  be the largest size of a set  $A$  mentioned in  $S_0$ . Construct a graph  $\mathcal{G}_0$  such that for every pair of disjoint finite sets  $A, B \subseteq V$  of sizes  $\leq n$  and  $\leq m$ , respectively, there is a vertex  $v \in V$  such that  $\mathcal{G}$  has edges between  $v$  and every element of  $A$ , but no edge between  $v$  and any element of  $B$ ; and such that for no set  $A$  of size  $n+1$ , there is a vertex  $v \in V$  such that  $\mathcal{G}$  has edges between  $v$  and every element of  $A$ ; then  $\mathcal{G} \models S_0$  but  $\mathcal{G} \not\models S$ .

**E2 ans.** Obviously,  $0 \leq |\mathcal{U}| \leq |\gamma|^{\aleph_0}$ , and  $|\mathcal{U}| = 0$  if  $\text{cf}(\gamma) > \omega$ . If  $\text{cf}(\gamma) = \omega$ , let  $f : \omega \rightarrow \gamma$  be strictly increasing and cofinal. Define an injection  $I$  from  ${}^\omega\gamma$  into  $\mathcal{U}$  (so that  $|\mathcal{U}| \geq |\gamma|^{\aleph_0}$ ) by:  $I(s) = \{\sum_{\ell \leq n} (f(\ell) + s(\ell)) : n \in \omega\}$ . Note that these finite ordinal sums lie below  $\gamma$  because  $\gamma = \omega^\delta$ .

**E3 ans.** Without knowing whether  $T$  is consistent, we can assume that  $T$  is a theory (closed under  $\vdash$ ). Fix a computable listing of  $T$  as  $\{\psi_n : n \in \omega\}$ . Define  $\varphi_e$  so that  $\varphi_e(n)$  is 0 unless  $\psi_n$  is the sentence  $0 = 1$ , in which case the computation goes into an infinite loop. Then  $\varphi_e$  is total iff  $T$  is consistent. So,  $\varphi_e$  is total, but, by the Incompleteness Theorem,  $T$  cannot prove this.

**C1 ans.** These are exactly the c.e. Turing degrees. One direction is trivial; for the other direction, fix an arbitrary c.e. set  $B$ , enumerate all elements of the form  $3^{n+1}$  into  $A$ , and when  $x$  enters  $B$  at stage  $s$ , enumerate  $3^{s+1} + 2^x$  into  $A$ .

**C2 ans.** Let  $\{T_s\}_{s \in \omega}$  be a computable sequence of approximations to  $T$ . We may assume that each  $T_s$  is a (finite) tree. Let  $h(n)$  be the least stage  $s \geq n$  such that  $A \upharpoonright (n+1)$  is on  $T_s$ . Then  $f \leq_T A$ , so there is a computable  $f$  majorizing  $h$ .

We define  $Q$  in levels. Let  $\sigma$  be in  $Q$  if its prefixes are in  $Q$  and if  $\sigma \in T_s$  for some  $s$  such that  $|\sigma| \leq s \leq f(|\sigma|)$ . So  $Q$  is a tree and, by the choice of  $f$ ,  $A \in [Q]$ .

To see that  $[Q] \subseteq [T]$ , assume that  $\tau \notin T$ . Then there is a stage  $n$  such that  $(\forall s \geq n) \tau \notin T_s$ . Each  $T_s$  is a tree, so the same holds for every  $\sigma$  extending  $\tau$ . In particular, no  $\sigma$  extending  $\tau$  of length  $n$  is on  $Q$ .

**C3 ans.**

- (a) Without loss of generality, you may assume that there is no greatest element (since the finitely many elements at the end can be deleted.)  
Now simply search for the next element of such a sequence.
- (b) Use a finite-injury priority argument to meet the requirements that no total function enumerates an infinite computable strictly ascending or descending sequence.