Instructions: **Do all six problems.**[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let κ be an infinite cardinal and let $\mathcal{A} \subseteq [\kappa]^{<\kappa}$ have cardinality κ. Prove that there is a 1-1 enumeration $\{A_\alpha : \alpha < \kappa\} = \mathcal{A}$ such that $|\bigcup_{\xi<\alpha} A_\alpha| < \kappa$ for all $\alpha < \kappa$.

As usual, $[\kappa]^{<\kappa} = \{x \subseteq \kappa : |x| < \kappa\}$. *Caution.* This problem is trivial when κ is regular.

E2. Prove that any decidable consistent L-theory T which is decidable complete consistent L-theory $T' \supseteq T$. Recall that T is decidable iff there is an algorithm which will decide for any L-sentence θ whether or not $T \vdash \theta$.

E3. Prove or disprove: there exists a partial computable function f such that the domain of f and range of f are not computable but the graph of f is computable.

C1. Let $f_0, f_1, f_2, \ldots$ be a sequence of functions on ω such that each f_i is hyper-immune relative to the join of the others (in other words, f_i is not dominated by any $(\bigoplus_{j \neq i} f_j)$-computable function). Show that there is a 1-generic computable from $\bigoplus_{j\in\omega} f_j$.

C2. Prove there exists a partial computable $\psi : \omega \to \omega$ such that the domain of ψ, $\mathrm{dom}(\psi)$, is co-infinite but for every partial computable ρ which extends ψ we have that $\mathrm{dom}(\rho) \setminus \mathrm{dom}(\psi)$ is finite. Can we have such a ψ with range $\{0, 1\}$?

C3. Let x be a incomputable real, find two Turing incomparable reals a and b such that $a + b = x$.

[1]Note that this is different from past exams.

Sketchy Answers or Hints

E1 answer. Assume that κ is a limit cardinal. There are two cases.

Case 1. For some cardinal $\gamma < \kappa$ the set $\mathcal{A} \cap [\kappa]^{<\gamma}$ has cardinality κ.

Case 2. Not case 1.

In case 2 list the family so that if $\alpha < \beta$ then $|A_\alpha| \leq |A_\beta|$. In case 1 construct a listing with $|A_\alpha| \leq |\alpha| + |\gamma|$ by "filling in" using the elements of $\mathcal{A} \cap [\kappa]^{<\gamma}$.

E2 answer. Use that $T \cup \{\theta\}$ is inconsistent iff $T \vdash \neg\theta$.

E3 answer. There is such an f. Let a_n be an effective 1-1 enumeration of a c.e. set which is not computable. Let f be the function whose graph is $\{(2n, 2a_n), (2a_n + 1, 2n + 1) : n < \omega\}$.

C1 answer. J Miller - result due to Damir and Adam.

We construct an infinite binary string A by finite initial segments σ_n for $n \in \omega$. The requirement is to force the jump (1-generic). At σ_n, for each $e < n$, in order to force $\varphi_e^A(e)$ to converge, we search for extensions of σ_n up to length $f_e(n)$ for convergence of the oracle computation $\varphi_e(e)$, and pick the highest priority e for which we find such extension and let it be σ_{n+1}.

So the construction is recursive in the join of all f_i's. Now if some jump-forcing requirement is not satisfied, say the least such is e_0, then it is easy to see that the construction is recursive in the join of all others since the jump-forcing requirement for e_0 has never acted. Therefore at each σ_n, the first extension which forces $\varphi_{e_0}(e_0)$ to converge is bounded by some length recursive in the join of all other f_i's. Then f_{e_0} being hyperimmune relative to the join of all others gives the desired contradiction.

C2 answer. Let a_n be a 1-1 effective enumeration of a simple set and define $\psi(a_n) = n$. To get such a ψ with range $\{0, 1\}$. Apply two theorems of Friedberg. Let A be a maximal set and let $A = A_0 \sqcup A_1$ be a splitting into c.e. non-computable sets. Define ψ by $\psi^{-1}(i) = A_i$.

C3 answer. Construct a and b by finite initial segments. Say we have α and β, in order to force $\varphi_e^a \neq b$. There are three subcases here:

1, if there is an n such that $\varphi_e^a(n)$ is always divergent, then we have forced φ_e^a to be partial.

2, if there are extensions of α and β which satisfy $\varphi_e^a(n) \downarrow\neq b(n)$ and $a + b = x$ is still possibly true, then we can take these extensions.

3, otherwise, then we can compute x by searching for long enough extensions of α which converge on long enough bits, which now agree with all possible b's which satisfy $a + b = x$. This allows us to limit x into smaller and smaller intervals and so compute longer and longer initial segments of x. Note that this algorithm will fail if x is a dyadic rational (say we are in base 2), but of course x is incomputable here.