Instructions: **Do all six problems.**[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

**E1.** Let $M$ model PA.

1. Show that there is no formula $\phi(x, y)$ so that every subset $D \subseteq M$ definable (with parameters) is defined in $M$ by $\phi(x, b)$ for some $b$.
2. Show that for every $c \in M$, there is a formula $\phi(x, y)$ so that every subset $D \subseteq [0, c)$ definable (with parameters) is defined in $M$ by $\phi(x, b)$ for some $b$.

**E2.** Ordinal addition and multiplication are not commutative. Generalize this as follows: Let $\star$ be any binary function on $\omega_1$. Assume that $\alpha \star 2 > \alpha$ for all $\alpha > \omega$, and that $\star$ is continuous in the sense that $\alpha \star \beta = \sup_{n\in\omega}(\alpha \star \beta_n)$ whenever $\beta_0 < \beta_1 < \beta_2 < \cdots$ and $\beta = \sup_{n\in\omega} \beta_n$. Prove that $\star$ is not commutative.

**E3.** Call a model $M$ "nice" iff for every $a, b \in M$, there is an automorphism of $M$ that moves $a$ to $b$. Let $T$ be a theory in a countable language. Show that if $T$ has a nice model of some infinite cardinality, then $T$ has nice models of all infinite cardinalities.

**C1.** Prove or disprove:

- The d.c.e. sets are closed under union.
- The d.c.e. sets are closed under intersection.

[1] Note that this is different from exams up until a year ago.

**C2.** Use a finite-injury argument to show that there exists a non-auto-reducible c.e. set. (Recall that a set $A$ is *autoreducible* if there is a Turing functional $\Phi$ such that for all $x$, $A(x) = \Phi^{A-\{x\}}(x)$.)

**C3.** Construct an infinite set $X$ and an infinite set $G$ disjoint from $X$ such that for every $Y \subseteq X$, $Y \cup G$ is 1-generic.

Sketchy Answers or Hints

**E1 answer.**

1. Consider the definable set $\{x \mid \neg\phi(x,x)\}$.
2. $\phi(x,y)$ states that $x \in D$ iff the $x$th prime divides $b$.

**E2 answer.** Fix a limit ordinal $\beta$ with $\omega < \beta < \omega_1$ such that $\beta$ is closed under $\star$. Say $\beta = \sup_{n\in\omega} \beta_n$ where $\omega < \beta_0 < \beta_1 < \beta_2 < \cdots$. Then $\beta \star 2 > \beta$, but each $2 \star \beta_n < \beta$, so $2 \star \beta \leq \beta$ by continuity.

**E3 answer.** Expand the language by adding a ternary function $A(x,y,z)$. The intent is that for each "fixed" $x, y$, $A(x,y,z)$ is an automorphism that moves $x$ to $y$.

To formalize this, add an axiom saying that for each $x, y$, the map $z \mapsto A(x,y,z)$ is a permutation of the model moving $x$ to $y$. Also, for each symbol of the language, add an axiom saying that this permutation is an automorphism with respect to that symbol. For example, if $P$ is three-placed predicate, add an axiom saying that for all $x, y$, and all $z_1, z_2, z_3, w_1, w_2, w_3$: $w_1 = A(x,y,z_1) \wedge w_2 = A(x,y,z_2) \wedge w_3 = A(x,y,z_3)$ implies that $P(z_1,z_2,z_3) \leftrightarrow P(w_1,w_2,w_3)$.

Then just apply the standard Löwenheim-Skolem Theorem to the new theory in the expanded language.

**C1 answer.**

- No: There is a properly 3-c.e. set $A$ (by a direct argument) which can be written as the union of a d.c.e. set $A_1 - A_2$ and a c.e. set $A_3$ for c.e. sets $A_1 \supset A_2 \supset A_3$.
- Yes: If $A = A_1 - A_2$ and $B = B_1 - B_2$ for c.e. sets $A_1$, $A_2$, $B_1$ and $B_2$, then $A \cap B = (A_1 \cap B_1) - (A_1 \cup A_2)$ is clearly d.c.e.

**C2 answer.** For each $e$, fix a witness $x$ and wait for $\Phi^{A-\{x\}}(x) = 0$. Then enumerate $x$ into $A$ and restrain the rest of $A$ on the use of $\Phi^{A-\{x\}}(x)$. Organize these strategies in a finite-injury argument.

**C3 answer.** We construct a sequence $Z \in \{0, 1, *\}^\omega$. We define $G$ by $G(n) = 1 \iff Z(n) = 1$ and $X$ by $X(n) = 1 \iff Z(n) = *$. In other words, we want $Z$ to be 1-generic no matter how we replace the $*$'s in $Z$ with 0's and 1's.

If $\sigma \in \{0, 1, *\}^{<\omega}$ has $n$ many $*$'s and $\rho \in 2^n$, let $\sigma[\rho]$ be the binary string that results from replacing the $*$'s in $\sigma$ with the bits from $\rho$.

We build $Z$ by initial segments. Let $\sigma_0$ be the empty string. Now assume that we have constructed $\sigma_n$ and that it has $n$ many $*$'s. Let $W_n$ be the next c.e. set of binary strings. We want to meet or avoid $W_n$ for every way that we can substitute for the $*$'s in $\sigma_n$. Let $\rho_1, \ldots, \rho_{2^n}$ list all of the binary strings of length $n$. Let $\tau_0 = \sigma_n$. Assume that we have defined $\tau_i$. If there is a $\mu$ such that $\tau_i[\rho_{i+1}]\mu \in W_n$, then let $\tau_{i+1} = \tau_i\mu$. Otherwise, let $\tau_{i+1} = \tau_i$. Let $\sigma_{n+1} = \tau_{2^n}*$. Finally, let $Z = \bigcup_{n\in\omega} \sigma_n$.