Instructions:

Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Consider a countable family $\mathcal{F} \subseteq 2^\omega$. We say that $A \subseteq \omega \times \omega$ is a *listing* of $\mathcal{F}$ if $\mathcal{F} = \{A^{[n]} \colon n \in \omega\}$, where $A^{[n]} = \{m \colon \langle n, m\rangle \in A\}$ is the nth column of A.

(a) Construct a graph $\mathcal{G}$ such that a Turing degree *enumerates* a listing of $\mathcal{F}$ if and only if it computes a presentation of $\mathcal{G}$.

(b) Construct a graph $\mathcal{H}$ such that a Turing degree *computes* a listing of $\mathcal{F}$ if and only if it computes a presentation of $\mathcal{H}$.

E2. Let $\mathcal{L}$ be an uncountable language for first-order logic consisting only of function symbols. Let $\mathfrak{A}$ be an uncountable structure for $\mathcal{L}$. If $X \subseteq A$, let $\mathrm{cl}(X)$ be the closure of X under all the functions of the model. Call $S \subseteq A$ *nice* iff there is a countable $X \subseteq S$ such that $S \subseteq \mathrm{cl}(X)$.

Assume that $S \subseteq A$ is not nice (so it is clearly uncountable). Prove that there is an uncountable $T \subseteq S$ so that no uncountable subset of T is nice.

E3. Let G be a computable Gödel numbering of all Σ_1 formulas in the language of PA, so every integer represents a Σ_1-formula. Define the equivalence relation on ω given by $G(\phi) \sim G(\psi)$ if $\mathrm{PA} \vdash \psi \leftrightarrow \phi$. We say that a c.e. equivalence relation is *precomplete* if it is non-trivial (i.e. there are two numbers which are not equivalent) and for any partial computable function f there is a total computable function g so that $f(n)\downarrow \Rightarrow f(n) \sim g(n)$. Show that $\sim$ is a precomplete c.e. equivalence relation.

[1]Note that this is different from exams up until two years ago.

Computability Theory

C1. Assume that a countable family $\mathcal{F} \subseteq 2^\omega$ contains all finite sets. Show that a Turing degree $\mathbf{d}$ computes a listing of $\mathcal{F}$ if and only if it computes a unique listing of $\mathcal{F}$, i.e., one in which each set in $\mathcal{F}$ appears exactly once.

C2. We say that a c.e. equivalence relation E is universal if whenever R is any other c.e. equivalence relation, there is a total computable function $h : \omega \to \omega$ so that $nRm \Leftrightarrow h(n)Eh(m)$.

a) Show that there is a universal c.e. equivalence relation.

b) Show that the set of e so that W_e is a c.e. equivalence relations which is universal is a Σ^0_3-complete set.

C3. Prove that for any $i \in \omega$ and any B such that $0' \leq_T B$, there exists A such that $A \oplus W_i^A \equiv_T A \oplus 0' \equiv_T B$.

Sketchy Answers or Hints

E1 ans. The graphs will be "daisy graphs". The connected components of a daisy graph are "daisies", which are graphs consisting of loops (the petals) that are disjoint except that they all share one point. Say that a daisy *codes* a set $B \subseteq \omega$ if it has exactly one loop of size $n+3$ for each $n \in B$, and no other loops. Note that it is equally hard to enumerate B as it is to compute a presentation of the daisy that codes B. For (a), let $\mathcal{G}$ be the daisy graph consisting of countably many daisies coding B for each $B \in \mathcal{F}$, and no other daisies. It is easy to see that a presentation of $\mathcal{G}$ allows us to enumerate a listing of $\mathcal{F}$, and vice versa. For (b), note that it is equally hard to compute B as it is to compute a presentation of the daisy that codes $B \oplus \overline{B}$, where $\overline{B}$ is the complement of B. So let $\mathcal{H}$ be the daisy graph consisting of countably many daisies coding $B \oplus \overline{B}$ for each $B \in \mathcal{F}$, and no other daisies.

E2 ans. Choose $t_\xi \in S$ for $\xi < \omega_1$ such that $t_\xi \notin \mathrm{cl}(\{t_\eta : \eta < \xi\})$. Then, let $T = \{t_\xi : \xi < \omega_1\}$.

E3 ans. Let $D(x, y)$ be a first-order formula defining the function which sends $G(\phi)$ to the (usual) Gödel code for $G^{-1}(f(G(\phi)))$. Then consider the function $g(n) = \exists m(G(D(n,m)) \wedge \mathrm{Tr}_{\exists_1}(m))$. Check this works.

C1 ans. The result is folklore. The proof sketch below is taken from "Generic Muchnik reducibility and presentations of fields" by Downey, Greenberg, and Miller.

This is a finite injury construction. Let B list the sets in $\mathcal{F}$, possibly with repetitions. We compute a unique listing A of the same collection of sets. At any given stage in the construction of A, only finitely many values (of finite columns) of the listing will have been determined. Uniqueness will be a global requirement. In addition, we have requirements of the form

$$R_n\colon (\exists m)\ A^{[m]} = B^{[n]}.$$

To meet R_0, we let $A^{[0]}$ copy $B^{[0]}$ and restrain lower priority strategies from affecting $A^{[0]}$. For $n > 0$, the strategy for R_n is initialized with a list

$A^{[0]}, \dots, A^{[r]}$ of columns of the listing A that are restrained by higher priority requirements. The strategy waits for a stage at which it sees that $B^{[n]}$ is different from how each of $A^{[0]}, \dots, A^{[r]}$ have been defined. Say that such a stage is found. The strategy for R_n acts as follows: Let m be large enough that $A^{[m]}$ is currently undefined on all values. The strategy declares that $A^{[m]}$ will copy $B^{[n]}$ (which will happen unless R_n is later injured). It restrains $A^{[0]}, \dots, A^{[m]}$ and reinitializes all lower priority requirements (ensuring that they will respect this restraint and injuring any that have already acted). Finally, the strategy declares each of $A^{[r+1]}, \dots, A^{[m-1]}$ to be distinct finite sets (hence in $\mathcal{F}$) different from each of $A^{[0]}, \dots, A^{[r]}$ and $A^{[m]}$.

C2 ans. Fix a computable enumeration $\{R_i\}_{in\omega}$ of all c.e. equivalence relations by making R_i the reflexive, symmetric, transitive closure of W_i.
a) E.g., define a universal c.e. equivalence relation $\langle i, x\rangle R \langle j, y\rangle$ by $i = j$ and xR_iy.
b) Let $I = \{x \mid R_x \text{ is universal}\}$. An easy calculation, using the fact that a ceer R is universal if and only if $E \le R$, for a fixed universal ceer E, shows that $I \in \Sigma^0_3$, namely,

$$x \in I \Leftrightarrow (\exists e)[\phi_e \text{ is total and } \phi_e \text{ reduces } E \text{ to } R_x].$$

Next, we show that for every $S \in \Sigma^0_3$, we have $S \le_m I$. Given S, fix a universal ceer E and a c.e. class $\{X_{\langle i,j\rangle} : i, j \in \omega\}$ such that

$$i \in S \Rightarrow (\exists j)[X_{\langle i,j\rangle} = \omega],$$
$$i \notin S \Rightarrow (\forall j)[X_{\langle i,j\rangle} \text{ finite}],$$

Uniformly in i, build a ceer R such that, denoting by $R^{[j]}$ the ceer

$$x \; R^{[j]} \; y \Leftrightarrow \langle j, x\rangle \; R \; \langle j, y\rangle,$$

we have that

$$i \in S \Rightarrow (\exists j)[R^{[j]} = E],$$
$$i \notin S \Rightarrow R \text{ yields a partition into finite sets.}$$

This is enough to prove the claim, since a universal ceer has always (infinitely many) infinite equivalence classes; indeed, if E, T are ceers such that $E \le T$

via a computable function f, and $[x]_E$ is an undecidable equivalence class, then so is $[h(x)]_T$, as $[x]_E = f^{-1}[[h(x)]_T]$.
Construction Let $\{E_s\}_{s\in\omega}$ be a computable approximation to E as a c.e. set, with each E_s finite, and consider a computable approximation $\{X_{\langle i,j\rangle,s}\}_{s\in\omega}$ to $\{X_{\langle i,j\rangle}\}_{i,j\in\omega}$ via finite sets: We say that $s+1$ is $\langle i,j\rangle$-*expansionary* if

$$X_{\langle i,j\rangle,s+1} - X_{\langle i,j\rangle,s} \neq \emptyset.$$

Stage by stage we define, uniformly in i, a finite set R^s so that, eventually, $R = \bigcup_s R^s$ is our desired ceer.
Stage 0 Let $R^0 = \emptyset$.
Stage $s+1$ Let j be the least number $\leq s$, if any, such that $s+1$ is $\langle i,j\rangle$-expansionary. Then carry out the following, with the understanding that if there is no such j, then only item (1) applies:

1. For every $k \neq j$, $k \leq s$, and $x \leq s$, let $\langle\langle k,x\rangle, \langle k,x\rangle\rangle \in R^{s+1}$.
2. Let $\langle\langle j,x\rangle, \langle j,y\rangle\rangle \in R^{s+1}$ for every $\langle x,y\rangle \in E_s$.

It is straightforward to verify that if $i \notin S$ then every j has only finitely many $\langle i,j\rangle$-expansionary stages, so the equivalence classes of R are finite, hence R is not universal. Otherwise, for the least j such that there are infinitely many $\langle i,j\rangle$-expansionary stages, we have that $R^{[j]} = E$, hence $E \leq R$, i.e., R is universal.

C3 ans. Proceed as in the proof of the Friedberg Jump Inversion Theorem, but instead of forcing conditions of the form

$$(\exists\sigma \subset A)[\{e\}^\sigma(e)\downarrow \vee (\forall\tau \supseteq \sigma)[\{e\}^\tau(e)\uparrow]],$$

use conditions of the form

$$(\exists\sigma \subset A)[e \in W_i^\sigma \vee (\forall\tau \supseteq \sigma)[e \notin W_i^\tau]].$$