**Instructions:**

**Do all six problems.**[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

**E1.** Let $T_0$ be the theory of rooted trees of height exactly 2. That is, every model of $T_0$ is an acyclic graph where there is a node $r$ so that every other node is distance $\leq 2$ from $r$, and there is some node of distance exactly 2 from $r$. Characterize all of the $\aleph_0$-categorical completions of $T$. (Recall that a theory is $\aleph_0$-categorical if it has exactly one countably infinite model up to isomorphism.)

**E2.** Prove that neither the class of connected graphs nor the class of disconnected graphs is axiomatizable in a first-order language.

**E3.** Suppose that $\Gamma$ is a map from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ such that $\Gamma(A) \subseteq \Gamma(B)$ whenever $A \subseteq B$. Show that there is a set $X$ so that $\Gamma(X) = X$.

[1] Note that this is different from exams up until two years ago.

# Computability Theory

**C1.** Show that the family $\{\{n\} \oplus F \mid F \text{ is finite and } W_n \neq F\}$ has no uniform c.e. enumeration.

**C2.** We say a set $X$ is auto-enumerable if $(\exists i)(\forall j)\ A = W_i^{(A \setminus \{j\}) \oplus \{j\}}$. Intuitively, this means that if you remove single elements $j$ from $A$, the resulting sets still uniformly enumerate $A$.

Show that there exists a co-c.e. set $Y$ that is not auto-enumerable.

**C3.** Let $\mathcal{L}$ be a computable linear order with the property that every computable ordinal $\alpha$ embeds in an order-preserving way into $\mathcal{L}$. Show that $\omega_1^{\mathrm{CK}}$ embeds in an order-preserving way into $\mathcal{L}$.

Sketchy Answers or Hints

**E1 ans.** For a height 2 tree $T$, define the function $f : \omega+1 \to \omega+1$ by $f_T(\alpha)$ is the number of nodes of distance 1 from $r$ that are connected to exactly $\alpha$ nodes of distance 2 from $r$. We show that the completions of the theory of rooted trees of height exactly 2 are those with some $k \in \omega$ so $f_T(m) = 0$ whenever $m > k$ and $m \in \omega$. Thus, given a function $f : \omega + 1 \to \omega + 1$ with $f|\omega$ having finite support contained in $[0, k]$, we define a theory $T_f$ saying the following axiom: There exists a root $r$ so that:

- Everyone has distance $\leq 2$ from $r$, and someone has distance 2 from $r$.
- IF $f_T(\omega) = 0$: There are no elements of distance 1 from $r$ which are connected to more than $k + 1$ elements.
- IF $f_T(\omega) \geq n$: There are $n$ elements of distance 1 from $r$ which are connected to more than $k + 1$ elements.
- IF $f_T(\omega) < n$: There are not $n$ elements of distance 1 from $r$ which are connected to more than $k + 1$ elements.
- For each $j \leq k$ and each $m \in \omega$ which is $\leq f(j)$, there are at least $m$ many elements of distance 1 from $r$ which are connected to exactly $j$ elements of distance 2 from $r$
- For each $j \in \omega$ and each $m \in \omega$ which is $> f(j)$, there are not $m$ elements of distance 1 from $r$ which are connected to exactly $j$ elements of distance 2 from $r$.

It is straightforward to see that the axioms suffice to define an $\aleph_0$-categorical theory. Thus, the theories $T_f$ for such $f$ are also complete (either $\Sigma_{n\in\omega} n f_T(n) < \omega$, in which case the theory has only 1 finite model or is has only infinite models and is $\aleph_0$-categorical, so use downwards skolem to see completeness). Now, to see these are the only $\aleph_0$-categorical completions, fix any model $\mathcal{M}$ of our base theory (height exactly 2 trees) which fails to have a bound $k$ as above. If $\mathcal{M}$ does not have infinitely many elements of height 1 which are connected to infinitely many elements of height 2, then we use compactness and downward skolem to build a larger countable model. If $\mathcal{M}$ has infinitely many elements of height 1 which are connected to infinitely many elements of height 2, then consider the structure $\mathcal{N}$ which you get by throwing out all of those elements of height 1 which are connected to infinitely many elements of height 2 and all the elements of height 2 connected to such elements of height 1. Now, again, apply compactness and downward skolem to build a countable elementary extension of $\mathcal{N}$. Now, if needed, do this countably many times. The resultant structure is exactly $\mathcal{M}$ showing that $\mathcal{M} \equiv \mathcal{N}$, and the theory of $\mathcal{M}$ is not $\aleph_0$ categorical.

**E2 ans.** Consider the structure $(\mathbb{Z}, S)$, and let $\mathcal{A}$ be any elementary extension of $(\mathbb{Z}, S)$. Then $A \equiv (\mathcal{Z}, S)$, but one is connected while the other is not.

**E3 ans.** Define sets $S_i$ by induction: $S_0 = \emptyset$. If $\lambda$ is a limit ordinal, define $S_\lambda = \bigcup_{\gamma<\lambda} S_\gamma$. Define $S_{\beta+1} = \Gamma(S_\beta)$. By induction, see that if $\alpha < \beta$, then $S_\alpha \subseteq S_\beta$. It is impossible for each containment at a successor stage to be proper all the way up to $\omega_1$, since $\omega$ is countable. Where this containment is equality, we have $S_\alpha = S_{\alpha+1} = \Gamma(S_\alpha)$.

**C1 ans.** Suppose otherwise that the family has uniformly c.e. enumeration $W$. We then define a computable function $f$ as follows: For each $i$, we let $\phi_{f(i)}$ be the function which searches for the first thing enumerated into $W$ of the form $\{i\} \oplus F$. Then, on an input $x$, $\phi_{f(i)}(x)$ converges if and only if $x \in F$. By the recursion theorem, there must be some $i$ so that $W_{f(i)} = W_i$. But $W_{f(i)}$ is an $F$ so that $\{i\} \oplus F$ is in the family, thus $W_{f(i)}$ cannot equal $W_i$, which is a contradiction.

**C2 ans.** Finite injury...

**C3 ans.** Consider the set of computable linear orders such that every hyperarithmetical subset has a least element and that embed into $\mathcal{L}$. Now use overspill to give a computable linear order that is not a computable ordinal, every hyperarithmetical subset of which has a least element, and that embeds into $\mathcal{L}$ in an order-preserving way. We know that the order type of any computable linear order that is not a well-order but which is hyperarithmetically well-ordered begins with $\omega_1^{\mathrm{CK}}$, so $\omega_1^{CK}$ also embeds into $\mathcal{L}$ in an order-preserving way.