Instructions: Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let T be a computably axiomatizable theory. Show that if T has finitely many complete extensions, then it is decidable.

E2. Prove that the transitive closure of a binary relation cannot be defined in finite structures. Specifically, assume that we have a language with $R(x, y)$. For any formula $\varphi(x, y)$, show that there is a finite model $\mathcal{A}$ where φ does not define the transitive closure of R.

E3. Call a total order *almost dense* if and only if it has no first or last element and there are no triples $x < y < z$ such that y is the only element between x and z. Prove that there are $2^{\aleph_0}$ non-isomorphic countable almost dense total orders.

[1] Note that this is different from exams before January 2014.

Computability Theory

C1. Construct two sequences of sets $\{A_i\}_{i\in\omega}$ and $\{B_i\}_{i\in\omega}$ such that

- $(\forall i)\ A_i \equiv_T B_i$,
- $\{A_i\}_{i\in\omega}$ is *strongly independent*, meaning that $(\forall i)\ A_i \not\leq_T \bigoplus_{j\neq i} A_j$, and
- $\{B_i\}_{i\in\omega}$ is *not* strongly independent.

C2. Show that the index set of computable linear orders with intervals of arbitrarily long finite length is Π^0_3-complete.

C3. Show that given any sets A and B and any non-empty Π^0_1 class, there are elements P and Q in the class such that if $X \leq_T A \oplus P$ and $X \leq_T B \oplus Q$, then $X \leq_T A$ and $X \leq_T B$.

Model Theory

M1. Prove that theory of $(\mathbb{N}, S)$ is not finitely axiomatizable, where S is the successor function.

M2. Fix any structure A and element $a \in A$. Show that $\mathrm{Th}(A)$ is ω-categorical if the theory $\mathrm{Th}(A, a)$ is ω-categorical. Here $\mathrm{Th}(A, a)$ is formed by adding a single constant to the language which names the element a.

M3. Let T be a countable consistent complete theory. Show that there is a model M so that if a and b are in M and satisfy the same 1-type, then there is a formula $\theta(x, y)$ so that $M \models \theta(a, b)$ and in every model of T, any pair satisfying θ realize the same 1-type.

Sketchy Answers or Hints

E1 ans. Since T has only finitely many complete extensions, each complete extension can be axiomatized by adding a single sentence to T. So to check if a sentence follows from T, check if it is in all complete extensions of T, but complete theories with computable axiomatizations are decidable, thus T is decidable.

E2 ans. Let A_n be the structure with $2n$ elements in an R-loop. That is $A_n = \{a_1, \ldots, a_{2n}\}$ and $a_1 R a_2 \cdots R a_{2n} R a_1$. Further, let A_n have two constants c and d which name a_1 and a_n respectively. Let B_n be the structure with two R-loops of size n. That is, $B_n = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$ and $a_1 R \cdots a_n R a_1$ and $b_1 R \cdots R b_n R b_1$. Now we want to show that the A_n are too similar to the B_n for any formula φ to systematically tell them apart. **One method:** Let $\mathcal{A}$ be any ultraproduct of the structures A_n and $\mathcal{B}$ be any ultraproduct of the structure B_n. Then $\mathcal{A}, \mathcal{B}$ are both models of the theory of a single bijective function with no loops. That is, $\mathcal{A}, \mathcal{B} \models \mathrm{Th}(\mathbb{Z}, S)$ (which is complete). Further, c and d are in different $\mathbb{Z}$-chains in both $\mathcal{A}$ and $\mathcal{B}$. But this means they have the same 2-type in both $\mathcal{A}$ and $\mathcal{B}$ (see that any two elements of infinite distance have the same 2-type in a model of $\mathrm{Th}(\mathbb{Z}, S)$ by the fact that any such pair is automorphic with any other such pair). But that means that $\mathcal{A} \equiv \mathcal{B}$. Thus for any formula φ, there are (cofinitely many) n so that $A_n \models \varphi \leftrightarrow B_n \models \varphi$. Thus no formula φ can uniformly distinguish between A_n and B_n. If φ defined the transitive closure of R, then A_n would satisfy $\varphi(c, d)$ for every n and B_n would satisfy $\neg\varphi(c, d)$ for every n, and we showed this is not the case. **Alternatively**, play the Ehrenfeucht-Fraisse game (playing single elements, not tuples at every turn) on the pair (A_n, B_n) and see that Eloise wins any game of length less than $\log(n)$. As $\log(n)$ goes to infinity, this shows that no formula φ can distinguish every A_n from every B_n.

E3 ans. For any subset S of the set $\mathbf{Q}$ of rationals, let L be the linear order obtained from $\mathbf{Q}$ by replacing each element of S by a chain of length 2 (and leaving each $q \in \mathbf{Q} - S$ as a single point).

C1 ans. Let $\{A_i\}_{i\in\omega}$ be the columns of a 1-generic. Let $B_i = A_i$ except that $B_i(0) = B_0(i-1)$; this ensures that $B_0 \leq_T \bigoplus_{j\neq 0} A_j$.

C2 ans. Fix any Π^0_3-set S. Using that Fin is a Σ^0_2-complete index set, fix a computable function f such that $x \in S$ iff $W_{f(x,y)}$ is finite for all y. Without loss of generality, we may assume that $W_{f(x,y)} \subseteq W_{f(x,y+1)}$ for all x and y. Now, at stage 0, start with a linear order $L_{x,0}$ obtained from $\mathbf{Q}$ by replacing each $x \in \mathbf{N}$ by a chain of length $y+1$. At every stage $s+1$, "densify" in $L_{x,s+1}$ the chain at y one more time by "adding points in between and to the left and right". So in the limit, if $W_{f(x,y)}$ is finite, the part of L_x at y will be finite of length $> y$, whereas it will be dense without endpoints otherwise.

C3 ans. Let U be a nonempty Π^0_1 class. Build $P = \bigcap U^A_e$ and $Q = \bigcap U^B_e$ where U^Z_e is a nested sequences of nonempty $\Pi^0_1(Z)$ classes with $U^Z_0 = U$. Given $e = (i_A, i_B)$ first try to force either $\Phi^{A\oplus X}_{i_A}$ to be not total for all $X \in U^A_{e+1}$ or $\Phi^{B\oplus X}_{i_B}$ to be not total for all X in U^B_{e+1}. If that does not work, then find some n and $Z \in \{A, B\}$ such that both $[\{\sigma : \Phi^{Z\oplus\sigma}_{i_Z}(n)\downarrow\rightarrow \Phi^{Z\oplus\sigma}_{i_Z}(n) = 0\}] \cap U^Z_e \neq \emptyset$ and $[\{\sigma : \Phi^{Z\oplus\sigma}_{i_Z}(n)\downarrow\rightarrow \Phi^{Z\oplus\sigma}_{i_Z}(n) = 1\}] \cap U^Z_e \neq \emptyset$. If no such n is found argue that $\Phi^{Z\oplus X}_{i_Z}$ is computable for all X in U^Z_e. If such an n is found use it to diagonalize.

M1 ans. Assume Δ is a finite axiomatization of $Th(\mathbb{N}, S)$ and let T be any axiomatization of $Th(\mathbb{N}, S)$. Since $T \models \Delta$ and Δ is finite, by compactness there is some $T_0 \subseteq T$ finite such that $T_0 \models \Delta$. So there must exist some finite $T_0 \subseteq T$ axiomatizing of $Th(\mathbb{N}, S)$. Taking T to be the specific axiomatization

1. S is an injective function
2. There is a unique element not in the image of S
3. For every element a, $S^n(a) \neq a$ (an axiom for each $n \in \mathbb{N}$, meaning there are no S-cycles)

and letting $T_0 \subseteq T$ be finite, we can satisfy T_0 by appending to $(\mathbb{N}, S)$ a large enough S-cycle, which results in a structure not modeling T. So there is no $T_0 \subseteq T$ finitely axiomatizing $Th(\mathbb{N}, S)$.

M2 ans. The restriction map $S_n^A(a) \to S_n^A(\emptyset)$ is surjective. Then the theory $Th(A,a)$ being ω-categorical implies $|S_n^A(a)| < \aleph_0$ for every natural n, which implies $|S_n^A(\emptyset)| < \aleph_0$ for every natural n, so $Th(A)$ is ω-categorical.

M3 ans. Proceed as in the omitting types theorem, but instead of trying to omit any given type, you are trying to separate the type of every two elements. That is, we have requirements of the form

$$S_{i,j} : \text{The elements } c_i \text{ and } c_j \text{ have different 1-types}$$

We proceed as in the usual Henkin construction to build a model where every element is named by a constant c_i. To handle the $S_{i,j}$ requirements, we simply try to find some formula $\varphi(x)$ so that we can add $\varphi(c_i) \wedge \neg\varphi(c_j)$. If we cannot do this consistently, then from the sinlge formula $\Gamma(c_i, c_j, \bar{c})$ we have committed to so far, we can extract the formula $\theta(c_i, c_j) := \exists \bar{y} \Gamma(c_i, c_j, \bar{y})$. This θ will have to imply that c_i, c_j realize the same 1-type.