Instructions:

Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Consider the partial order $\mathbb{P} = (\mathcal{P}(\omega), \subseteq)$. Show that:

1. If α is an ordinal that order-embeds into $\mathbb{P}$, then α is countable.

2. $\mathbb{R}$ order-embeds into $\mathbb{P}$.

E2. Consider the sets

$$A = \{\ulcorner\varphi\urcorner : \mathrm{PA} \vdash \varphi\},$$
$$B = \{\ulcorner\varphi\urcorner : \mathrm{PA} \vdash \neg\varphi\},$$

where $\ulcorner\varphi\urcorner$ is the Gödel code of the sentence φ. Show that A and B are computably inseparable. I.e., show that there is no computable set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

E3. Show that the collection of free abelian groups (i.e., groups of the form $\oplus_{i \in I}\mathbb{Z}$) is not elementary.

[1] Note that this is different from exams before January 2014.

Model Theory

M1. Let G be an infinite simple group and let κ be infinite and less than the size of G. Show that G has a simple subgroup of size κ.

M2. Given $A \subseteq M$ and $b \in M$, show that the following are equivalent:

1. There is a definable function f and a tuple $\bar{a} \in A$ so that $b = f(\bar{a})$.
2. For every $N \succeq M$ and every automorphism σ of N, if σ fixes A pointwise, then it fixes b as well.

M3. Characterize all $\aleph_0$-categorical theories of a single unary 1–1 function.

Sketchy Answers or Hints

E1 ans.

1. Let f be an embedding form an ordinal α into $\mathbb{P} = (\mathcal{P}(\omega), \subseteq)$. We may assume (without loss of generality) that α is a limit ordinal. Let
$$g(\beta) = \min\{f(\beta+1) \setminus f(\beta)\}.$$
Note that $g\colon \alpha \to \omega$ is an injection, hence α is countable.

2. Fix a bijection between ω and $\mathbb{Q}$. Map $x \in \mathbb{R}$ to $\{q \in \mathbb{Q} \colon q \leq x\}$, i.e., its left cut in the rationals.

E2 ans. Use The Gödel Fixed Point Lemma. Suppose that C is a computable separator for A and B. Then let $\psi(x)$ be the formula that defines C. That is, for every x, $PA \vdash \psi(x)$ if and only if $x \in C$. Then use the Gödel fixed point lemma to get a formula φ so that $PA \vdash \varphi \leftrightarrow \neg\psi(\ulcorner\varphi\urcorner)$.

E3 ans. Consider the partial type $p(c) :=$ "c is divisible by every natural number". By compactness, c is consistent with the theory of free abelian groups, but cannot be realized in any free abelian group.

M1 ans. We argue that an elementary substructure of a simple group is itself a simple group. For any two $a, b \in G \setminus \{e\}$, by simplicity there is some way to generate b by applying conjugation and multiplication to a and a^{-1}, thus, if $a, b \in G_0 \preceq G$ then this fact is still true (quantifying with an existential on each conjugation) in G_0. So $G_0 \preceq G$ has no non-trivial normal subgroups, i.e., simple.
Now find the simple subgroup by applying downward Löwenheim-Skolem.

M2 ans. $2 \Rightarrow 1$: We argue that $\{b\}$ is defiable over A. Otherwise, the type $p(y) = tp^M(b/A) \cup \{y \neq b\}$ is finitely satisfiable, and so is realized by some c in some $N_0 \succeq M$. Since $tp^{N_0}(b/A) = tp^{N_0}(c/A)$, there exist $N_1 \succeq N_0$ and an automorphism f of N_1 fixing A with $f(b) = f(c)$, contradicting 2. Then

p(y) is not finitely satisfiable, meaning there is $\varphi(\bar{a}, y) \in tp^M(b/A)$ with b its unique solution in M. Define the formula $f(\bar{x}, y)$ to state that y is the unique element such that $\varphi(\bar{x}, y)$ holds (if precisely one exists), or otherwise $y = x_1$, where x_1 is the first element in $\bar{x}$.
1 $\Rightarrow$ 2: For an automorphism σ of some $N \succeq M$ fixing $\bar{a}$ and a $\emptyset$-definable function $f(\bar{x})$,

$$\sigma(b) = \sigma(f(\bar{a})) = f(\sigma(\bar{a})) = f(\bar{a}) = b$$

M3 ans. Suppose first that there is some element x such that $f^n(x) \neq x$ for all $n > 0$. Then any 2-type containing the formulas $f^n(x) \neq f^m(x)$ for any $n, m \geq 0$ cannot be isolated (and some such 2-type is consistent with the theory), contradicting Ryll-Nardzewski. A similar argument using Ryll-Nardzewski shows that in fact there is a fixed bound n_0 such that for any element x, $f^n(x) = x$ for some n with $0 < n \leq n_0$. These are the only limitations, i.e., any theory specifying for a finite number of cycle sizes that there are such and such finite number, or infinitely many, cycles of that size is $\aleph_0$-categorical.