Instructions: Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Say that we expand first-order logic by adding second-order quantifiers that range over finite subsets of the universe. (Incidentally, these are called *weak monadic second-order quantifiers.*) Show that the resulting logic does not admit compactness.

E2. Let T_0 and T_1 be computably enumerable, consistent extensions of PA (although, $T_0 \bigcup T_1$ need not be consistent). Show that there is a sentence ψ that is independent of both T_0 and T_1.

E3. Call a subset X of the plane *simple* if and only if X is closed and either X or its inverse X^{-1} is the graph of a monotonically non-increasing or non-decreasing function. (Here, $X^{-1} = \{(y, x) : (x, y) \in X\}$ and X is monotonically non-increasing iff $\forall (x_1, y_1), (x_2, y_2) \in X\ [x_1 < x_2 \to y_1 \geq y_2]$.) Prove that there is a subset L of the plane of size $2^{\aleph_0}$ that meets each simple set in a set of size $< 2^{\aleph_0}$. Work in ZFC.

[1]Note that this is different from exams before January 2014.

Model Theory

M1. Let T be a complete theory with infinite models. Suppose T has some model with an automorphism σ of fixed order $n > 1$. Let $\mathfrak{A}$ be any model of T. Show that there is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ which has an automorphism of order n.

M2. Let $\mathfrak{M}$ be a saturated structure. Suppose X is a definable (with parameters) subset of $|\mathfrak{M}|^n$. Suppose also that X is fixed by every automorphism of $\mathfrak{M}$.

a. Show that X is definable without parameters.

b. Explain why saturation is necessary.

M3. Let T be a complete theory and let $\Gamma(x)$ be a partial type. Suppose that in every model there are only finitely many realizations of $\Gamma(x)$. Show that the number of realizations of $\Gamma(x)$ is the same in every model.

Sketchy Answers or Hints

E1 ans. We can define that a model is finite (contradicting compactness), stating that there is a finite subset of the model containing all of its elements.

E2 ans. Call disjoint c.e. sets A and B *computably inseparable* if there is no computable set C such that A is a subset of C and B is disjoint from C. (Such a set C is called a *separator*.)

Given a c.e. set A, there is a Σ_1-formula φ_A in the language of arithmetic such that $n \in A$ if and only if PA $\vdash \varphi_A(n)$. (Note that it's not true, in general, that PA $\vdash \neg\varphi_A(n)$ for $n \notin A$. Indeed, this would mean that A is computable.)

Let A and B be computably inseparable c.e. sets. Let φ_A and φ_B be the corresponding formulas; more precisely, we modify $\varphi_A(n)$ to say that there is a witness that $n \in A$ which is $\leq$ the least witness that $n \in B$. Similarly, we modify $\varphi_B(n)$ to say that there is a witness that $n \in B$ which is $<$ the least witness that $n \in A$. Since A and B are disjoint, these modifications don't seem like they would do anything, but now we have:

PA $\vdash \varphi_A(n) \to \neg\varphi_B(n)$ and, equivalently, PA $\vdash \varphi_B(n) \to \neg\varphi_A(n)$.

Now let T_0 and T_1 be c.e. consistent extensions of PA. Such extensions can make new formulas of the form $\varphi_A(n)$ and $\varphi_B(n)$ true. Let A_0 be the set of n such that $T_0 \vdash \varphi_A(n)$. Define A_1, B_0, and B_1 similarly. Since PA $\vdash \varphi_B(n) \to \neg\varphi_A(n)$, we know that for every $n \in B_0$, hence every $n \in B$, $T_0 \vdash \neg\varphi_A(n)$. The same holds for T_1.

Case 1: There is an n such that $\varphi_A(n)$ is independent of both T_0 and T_1. So we're done.

Case 2: No such n exists. We will get a contradiction in this case. Define a computable set C as follows. To decide if $n \in C$, enumerate all proofs from T_0 and T_1 until one of the theories is first seen to prove either $\varphi_A(n)$ or $\neg\varphi_A(n)$. This must happen eventually because we are in case 2. If we see a proof of $\varphi_A(n)$, we put $n \in C$. Otherwise, $n \notin C$.

Now note that C is a computable superset of A: If $n \in A$ then both theories prove $\varphi_A(n)$, hence can't prove $\neg\varphi_A(n)$. It's also disjoint from B: If $n \in B$ then both theories prove $\neg\varphi_A(n)$, hence neither can prove $\varphi_A(n)$. Therefore, C is a computable separator of A and B, which cannot exist.

E3 ans. Let $\mathfrak{c} = 2^{\aleph_0}$. List all simple sets as $\{X_\alpha : \alpha < \mathfrak{c}\}$. Let $L = \{\vec{v}_\alpha : \alpha < \mathfrak{c}\}$, where $\vec{v}_\alpha \notin \{\vec{v}_\xi : \xi < \alpha\} \cup \bigcup\{X_\xi : \xi < \alpha\}$. To see that such a $\vec{v}_\alpha$ always exist, observe that if X is simple and the graph of a function, then X meets all but countably many horizontal lines in no more than one point and (obviously) then X meets every vertical line in no more than one point.

M1 ans. Let $\mathfrak{A}$ be a model with an automorphism σ of order $n > 1$. Let $\mathfrak{B}$ be a $|\mathfrak{A}|$-saturated elementary extension of $(\mathfrak{C}, \sigma)$. Then the reduct of $\mathfrak{B}$ to the original language is still $|\mathfrak{A}|$-saturated, so $|\mathfrak{A}|$-universal, so there is an elementary embedding of $\mathfrak{A}$ into it. (This can also be done in one line by saying "Take $\mathfrak{B}$ to be a resplendent elementary extension of $\mathfrak{A}$".)

M2 ans. Let $X = \varphi(\mathfrak{M}, a)$. Let $p = tp(a)$. By homogeneity of a saturated model, there are not two tuples b, c so that $tp(b) = tp(c) = p$ and $\varphi(\mathfrak{M}, b) \neq \varphi(\mathfrak{M}, c)$. By saturation, the partial type saying $tp(b) = tp(c) = p$ and $\varphi(\mathfrak{M}, b) \neq \varphi(\mathfrak{M}, c)$ is inconsistent. By compactness, there is a single $\psi \in p$ so that $\psi(b) \wedge \psi(c) \wedge \varphi(\mathfrak{M}, b) \neq \varphi(\mathfrak{M}, c)$ is inconsistent. Let $\rho(x) = \exists y \psi(y) \wedge \varphi(x, y)$. Then ρ defines X. — To see that saturation is necessary, consider in the language of a single equivalence relation E the countable structure $\mathfrak{N}$ that for each $n \in \mathbb{N} \cup \{\aleph_0\}$, there is a unique equivalence class of cardinality n. Each equivalence class is definable with parameters and preserved set-wise by every automorphism of $\mathfrak{N}$. However, the one infinite class is not definable without parameters.

M3 ans. Use compactness to show that Γ contains an algebraic formula. The number of realizations of an algebraic formula is determined by T. Let $\varphi(x)$ be an algebraic formula of minimal size (i.e. the least number of realizations). Show that φ isolates Γ, thus the number of realizations of Γ is the same as the number of realizations of φ, which is determined by T.