Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. (Work in ZF, i.e., **without the axiom of choice**.) Show the following:

1. There is a function mapping $\mathcal{P}(\omega)$ *onto* ω_1.
2. If $\mathcal{P}(\omega)$ is a countable union of countable sets, then $\mathrm{cf}(\omega_1) = \omega$.

E2. Let M be a model of PA that is not elementarily equivalent to $(\mathbb{N}, +, \cdot)$. Show that there is an infinite element of M that is definable.

E3. Let $\mathcal{C}$ be a class of L-structures (for some signature L) defined as follows: there is a set T of L-formulas with free variables among $\{x_1, \ldots, x_k\}$ such that if $\mathfrak{A}$ is an L-structure, then $\mathfrak{A} \in \mathcal{C}$ if and only if there is a tuple $\vec{a} \in A^k$ such that for every $\varphi \in T$ we have that $\mathfrak{A} \models \varphi[\vec{a}]$. Prove that if $\mathcal{C}$ is elementary, then it is axiomatized by the collection of sentences of the form $(\exists x_1 \ldots x_k) \bigwedge_{\varphi \in T'} \varphi$, where T' ranges over finite subsets of T.

Computability Theory

C1. A pair of disjoint c.e. sets A and B are *effectively inseparable* if there is a partial computable function ψ (called a *productive function* for the pair) such that for every pair of c.e. indices u, v,

$$A \subseteq W_u, B \subseteq W_v, \text{ and } W_u \cap W_v = \emptyset \Rightarrow \psi(u,v)\downarrow \text{ and } \psi(u,v) \notin W_u \cup W_v.$$

Show that every effectively inseparable pair has a total productive function.

An infinite set X is *r-cohesive* (*recursively cohesive*) if for every computable set C, either $X \subseteq^* C$ or $X \subseteq^* \overline{C}$.

C2. Prove that if X is r-cohesive, then it has hyperimmune degree (i.e., X computes a function that is not dominated by any computable function).

C3. Prove that if D is high (i.e., $D' \geq_T \emptyset''$), then D computes an r-cohesive set. *Hint.* Since D is high, there are $\Delta^0_2[D]$ approximations to the sets $\{e : \varphi_e \text{ is total and 0–1 valued}\}$ and $\{e : (\exists^\infty n)\, \varphi_e(n) = 1\}$.

Sketchy Answers or Hints

E1 ans.

1. Since $\omega \approx \omega \times \omega$, we have $\mathcal{P}(\omega) \approx \mathcal{P}(\omega \times \omega)$. So it is enough to define a surjective function $h\colon \mathcal{P}(\omega \times \omega) \to \omega_1$. Fix $R \in \mathcal{P}(\omega \times \omega)$. If (ω, R) is a well-order, let $h(R)$ be its order-type. Otherwise, let $h(R) = 0$. Every countable well-order is of the form (ω, R) for some $R \in \mathcal{P}(\omega \times \omega)$, so h is surjective.

2. Suppose $\mathcal{P}(\omega) = \bigcup_{i \in \omega} X_i$, where each X_i is countable. Let $f\colon \mathcal{P}(\omega) \to \omega_1$ be the surjective function that we proved to exist in the first part. There are two cases. If some $f[X_i]$ is cofinal in ω_1, then it witnesses that $\mathrm{cf}(\omega_1) \le \omega$. Otherwise, each $\alpha_i = \sup_{n \in X_i} f(n)$ is less than ω_1. But then $\{\alpha_i\}_{i \in \omega}$ is countable and witnesses that $\mathrm{cf}(\omega_1) \le \omega$. (It is clear that $\mathrm{cf}(\omega_1) \ge \omega$.)

E2 ans. Let φ be a formula (in prenex normal form) of lowest quantifier-complexity so that $M \models \varphi$ and $\mathbb{N}$ does not. We observe that φ must begin with an $\exists$. In particular, φ cannot begin with a $\forall$. Otherwise, $\mathbb{N} \models \neg\varphi$, and $\neg\varphi = \exists x \psi$ where ψ is of lower quantifier-complexity. But then $\mathbb{N} \models \psi(x)$ for some x. Let $\hat{x} = 1 + 1 + \cdots + 1$ (i.e. the term which represents the element x). Then $\mathbb{N} \models \psi(\hat{x})$. But then this is a sentence of lower quantifier-complexity than φ, and thus $M \models \psi(\hat{x})$. Thus $M \models \varphi$. So, φ must be $\exists_n$ for some n. Let $\varphi = \exists x \psi$. Let $a \in M$ be the least witness for ψ. The induction axioms in PA give us that there is a least witness. This witness is definable. We need only conclude that it is infinite. Suppose towards a contradiction that x is finite. Then x is represented by a term $\hat{x} = 1 + 1 + \cdots + 1$. But then $M \models \psi(\hat{x})$. Since ψ is of lower quantifier-complexity than φ, we can conclude that $\mathbb{N} \models \psi(\hat{x})$, so $\mathbb{N} \models \varphi$, a contradiction.

E3 ans. First note that if $\varphi(x_1, \ldots, x_k)$ is an L-formula, ψ is an L-formula, and $c_1, \ldots, c_k$ are constants not in L, then $\varphi(c_1, \ldots, c_k) \models \psi$ if and only if $(\exists x_1 \cdots \exists x_k)\ \varphi(x_1, \ldots, x_k) \models \psi$. Let $T(\vec{c}) = \{\varphi(x_1/c_1, \ldots, x_k/c_k) : \varphi \in T\}$, where $c_1, \ldots, c_k$ are new constants. The L-reducts of models of $T(\vec{c})$ are in $\mathcal{C}$, hence if ψ is a sentence that is true in all models from $\mathcal{C}$ then $T(\vec{c}) \models \psi$. By compactness there is a finite set $T' \subseteq T$ such that $T'(\vec{c}) \models \psi$, or equivalently $(\exists x_1 \cdots \exists x_k)\ \bigwedge_{\varphi \in T'} \varphi \models \psi$.

C1 ans. Let f and g be computable functions such that $W_{f(u)} = W_u \cup A$ and $W_{g(v)} = W_v \cup B$. Now wait until $\psi(f(u), g(v))$ converges or $W_{f(u)} \cap W_{g(v)} \neq \emptyset$, one of which must occur by assumption. Then set $p(u, v)$ to equal $\psi(f(u), g(v))$ or 0, respectively.

C2 ans. We prove the contrapositive. Consider the X-computable function g such that $g(n)$ is the least element of X that is $\geq n$. If X does not have hyperimmune degree, then there is a computable function f that majorizes g (i.e., $(\forall n)\ g(n) \leq f(n)$). Now let $F(0) = 0$ and $F(n+1) = f(F(n)) + 1$. So for each n, there is an element of X in the interval $[F(n), F(n+1))$, namely $g(F(n))$. Let

$$C = \bigcup_{n \in \omega} [F(2n), F(2n+1))\,.$$

It should be clear that C is computable, but that $X \cap C$ and $X \cap \overline{C}$ are both infinite. Hence X isn't r-cohesive.[1]

C3 ans. First, we construct $\Delta^0_2[D]$ approximations to indices $e_0, e_1, e_2, \ldots,$ such that

- φ_{e_i} is the characteristic function of an infinite computable set X_i,
- $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots,$
- If φ_i is the characteristic function of a computable set C, then either $X_i \subseteq C$ or $X_i \subseteq \overline{C}$.

To find $e_{i,s}$ at stage s, we assume that we have already determined $e_{i-1,s}$ (where e_{-1} is a fixed index for the characteristic function of ω). If we are not currently guessing that φ_i is total and 0–1 valued, then let $e_{i,s} = e_{i-1,s}$. Otherwise, check our current guess as to whether $X_{i-1,s} \cap C$ is infinite, where

[1] Actually, if X is r-cohesive, then X is a hyperimmune *set*. This is because we can compute a function that majorizes g from any function that majorizes the principal function of X.

φ_i is the characteristic function of C and $\varphi_{e_{i-1,s}}$ is the characteristic function of $X_{i-1,s}$. If so, let $e_{i,s}$ be an index of the characteristic function of $X_{i-1,s} \cap C$. Otherwise, let $e_{i,s}$ be an index for $X_{i-1,s} \cap \overline{C}$. Note that for all i, as long as we choose indices consistently, $e_i = \lim_{s \to \infty} e_{i,s}$ exists. These indices clearly satisfy our requirements.

Now, we are ready to define the r-cohesive set X. For each s, search for a stage $t \geq s$ and an $n \geq s$ such that for all $i \leq s$, either $\varphi_{e_{i,s},t}(n) \downarrow = 1$ or $e_{i,s} \neq e_{i,t}$. Note that this search must be successful. Put n into X. Note that $X \leq_T D$ because n cannot be put into X after stage n of the enumeration of X. Also note that $X \subseteq^* X_i$ because all of our guesses eventually stabilize. So if C is computable with characteristic function φ_i, then either $X \subseteq^* X_i \subseteq C$ or $X \subseteq^* X_i \subseteq \overline{C}$.