Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Show that the complement of an elementary class $\mathcal{C}$ is elementary if and only if $\mathcal{C}$ is finitely axiomatizable.

E2. Show that $<$ is not definable in $(\mathbb{N}, 0, S)$, where S is the successor function on $\mathbb{N}$.

E3. Show that there are at least continuum many pairwise non-isomorphic densely ordered subsets of $\mathbb{R}$ without end-points.

Computability Theory

C1. A set X is *CEA* (*c.e. in and above*) if there is a $Y <_T X$ (in particular, $Y \not\geq_T X$) such that X is Y-c.e. Prove that every 1-generic is CEA. *Hint.* Consider the set of pairs $\langle i, j\rangle$ where i lies in the 1-generic but the pair itself does not.

C2. An oracle L is *peculiar*[1] if there is an $\emptyset'$-computable function g such that if Φ_e^L is total computable then $\varphi_{g(e)} = \Phi_e^L$ (no assumption is made on $g(e)$ if Φ_e^L is partial or not computable).

1. Prove that every peculiar set is low.
2. Prove that there is a noncomputable peculiar set.

C3. A relation $R \subseteq \omega^2$ is a *ceer* (*c.e. equivalence relation*) if it is c.e. and an equivalence relation. Prove that there is a ceer with infinitely many classes but with no c.e. infinite set of pairwise inequivalent elements.

[1] Not standard nomenclature.

Model Theory

M1. Suppose M is countable and every countable elementary extension of M is isomorphic to M. Show that M must be saturated.

M2. Show that no infinite field is $\aleph_0$-categorical.

M3. Let $\mathcal{L}$ be a relational language. Let φ be a universal $\mathcal{L}$-sentence.

a) Show that if φ has an infinite model, then φ has a model $\mathcal{M}$ with universe $\{a_n \mid n \in \omega\}$ so that $\langle a_n \mid n \in \omega \rangle$ is an indiscernible sequence in $\mathcal{M}$.

b) Let $\varphi := \forall x_1 \ldots \forall x_n \psi$ where ψ is quantifier-free. Show that φ has an infinite model if and only if φ has a model $\mathcal{N} = \{a_1, \ldots, a_m\}$ of size $\geq n$ so that $\langle a_1, \ldots, a_m \rangle$ is atomic-indiscernible. That is, for every relation symbol $R \in \mathcal{L}$, and tuples of indices $(i_1, \cdots, i_k)$ and $(j_1, \cdots, j_k)$ of the same order-type,

$$\mathcal{N} \models R(a_{i_1}, \ldots, a_{i_k}) \iff \mathcal{N} \models R(a_{j_1}, \ldots, a_{j_k}).$$

c) Let $\mathcal{L}$ be a recursive language. Deduce that the set of universal $\mathcal{L}$-formulas φ which have an infinite model is a recursive set.

Sketchy Answers or Hints

E1 ans. If $\mathcal{C}$ can be axiomatized by a finite set, then it can be axiomatized by the conjunction φ of the sentences in the finite set, and so the complement of $\mathcal{C}$ is axiomatized by $\{\neg\varphi\}$. — Conversely, suppose the complement of $\mathcal{C}$ is axiomatized by a set B; and that $\mathcal{C}$ is axiomatized by a set A. Then $A \cup B$ is inconsistent, so by compactness contains a finite inconsistent subset $A_0 \cup B_0$ with $A_0 \subseteq A$ and $B_0 \subseteq B$. Then A_0 axiomatizes $\mathcal{C}$; else there is a model satisfying A_0 in the complement of $\mathcal{C}$ which therefore also satisfies B_0, a contradiction.

E2 ans. Suppose the formula $\varphi(x, y)$ defines $<$ in $(\mathbb{N}, 0, S)$. By compactness, there is a model $\mathcal{M}$ of $\mathrm{Th}(\mathbb{N}, 0, S)$ containing two elements a and b such that for all $n \in \mathbb{N}$, $S^n(0) \neq a, b$; $S^n(a) \neq b$; and $S^n(b) \neq a$. Now in $(\mathbb{N}, 0, S)$ and thus also in $\mathcal{M}$, every element has an S-successor, and every element except for 0 has an S-predecessor; so in $\mathcal{M}$, both a and b are contained in distinct $\mathbb{Z}$-chains given by S. Without loss of generality, assume $\mathcal{M} \models \varphi(a, b)$, so there is an expansion of $\mathcal{M}$ in which $a < b$. Let f be the bijection on M mapping a to b and vice versa, extending this to a map swapping the $\mathbb{Z}$-chains of a and b, and leaving all other elements of M fixed. Then f is a $\{0, S\}$-isomorphism of $\mathcal{M}$ which reverses the order of a and b, a contradiction.

E3 ans. We code an arbitrary subset $S \subseteq \mathbb{N}$ into a dense subset of $\mathbb{R}$ as follows: Let $I = \{0\} \cup \{x \in (0, 1] \mid 2^{-(2n+1)} \leq x \leq 2^{-2n} \implies x \in \mathbb{Q}\}$, and let $J = \{1 - x \mid x \in I\}$, so both I and J are dense subsets of $[0, 1]$. Then the order type of

$$R_S = \mathbb{R}^- \cup \left(\bigcup_{m \in S} \{m + x \mid x \in I\} \right) \cup \left(\bigcup_{m \notin S} \{m + x \mid x \in J\} \right)$$

uniquely determines S, and each R_S is densely ordered without endpoints.

C1 ans. Fix a pairing function $\langle \cdot, \cdot \rangle$ which is computable, satisfies $i, j < \langle i, j \rangle$, and has computable coinfinite range. (For example we could define $\langle i, j \rangle = 2^i 3^j$.) Let G be a 1-generic. Define $Y := \{\langle i, j \rangle \notin G \mid i \in G\}$.

Clearly Y is computable in G. To show that G is Y-c.e., observe that $i \in G$ if and only if there is some j such that $\langle i, j \rangle \in Y$. (By immunity of G, for each i, $\{\langle i, j\rangle \mid j \in \omega\}$ intersects the complement of G.) To show that G is not computable in Y, think about how one would construct G in order to ensure that Y (defined from G as above) does not compute G. For each $\sigma \in 2^{<\mathbb{N}}$, define $\sigma' \in 2^{|\sigma|}$ by $\sigma'(m) = 1$ if and only if $m = \langle i, j\rangle$ for some i, j, $\sigma(m) = 0$, and $\sigma(i) = 1$. Observe that if σ is an initial segment of G, then σ' is an initial segment of Y. Assume towards a contradiction that $\Phi_e^Y = G$. Consider the recursive set S of σ such that $\Phi_e^{\sigma'}$ and σ are incompatible. Since $\Phi_e^Y = G$, no initial segment of G lies in S. By genericity, there is some initial segment τ of G with no extension in S. But we can extend τ to a string in S as follows. By immunity of G, there is some $n > |\tau|$ such that n does not lie in the range of the pairing function and $n \notin G$. Since $\Phi_e^Y = G$, there is some σ extending τ such that $\Phi_e^{\sigma'}(n) \downarrow = \sigma(n) = 0$. We define a string $\rho \in 2^{|\sigma|}$ such that $\rho(n) = 1$ and $\rho' = \sigma'$ by iteratively flipping certain 0s in σ as follows. Begin by defining $\rho(n) = 1$. In order to ensure that $\rho'(\langle n, j\rangle) = 0$ $(= \sigma'(\langle n, j\rangle))$, we define $\rho(\langle n, j\rangle) = 1$ for each j. In general, whenever we define $\rho(m) = 1 \neq 0 = \sigma(m)$, we will define $\rho(\langle m, j\rangle) = 1$. One can show that $\rho' = \sigma'$ and ρ extends τ. Hence ρ is an extension of τ which lies in S, contradiction.

C2 ans.

1. Let L be peculiarly low as witnessed by g. Using the parameter theorem define a total computable function f so that $\Phi^L_{f(e)}(s) = 0$ if $e \notin L'_s$ and 1 otherwise. $\emptyset'$ can answer whether $\varphi_{g(f(e))}(s) = 1$ for some s.

2. Build $L = \bigcup_n \sigma_n$ using a $\emptyset'$-computable initial segment construction: suppose that we have built σ_n and let $n = (e, i)$. Search for $\tau_0, \tau_1 \succeq \sigma_n$ such that for some m we have $\Phi_e^{\tau_0}(m) \downarrow \neq \Phi_e^{\tau_1}(m) \downarrow$. Define σ_{n+1} as τ_0 or τ_1 ensuring $\varphi_i(m) \neq \Phi_e^{\sigma_{n+1}}(m)$ (if $\varphi_i(m)$ is at all defined). If there are no such τ_0 and τ_1 then define $\sigma_{n+1} = \sigma_n$. This is a $\emptyset'$-computable construction, so if Φ_e^L is computable, then $\emptyset'$ can find the least i such that at stage (e, i), no splitting was found and then compute an index for Φ_e^L given $\sigma_{(e,i)}$.

C3 ans. For each e, let Q_e be the requirement that either W_e is finite or W_e contains distinct numbers which are equivalent. Arrange the requirements

$Q_0, Q_1, \ldots$ in order of priority. We begin the construction with the empty ceer, i.e., each number is in its own equivalence class. At stage s, a requirement Q_e is satisfied if W_e contains distinct numbers which are equivalent. A requirement Q_e requires attention if it is not satisfied and there are distinct numbers $x, y \in W_{e,s}$ whose equivalence classes are not restrained by requirements of higher priority. We act for the requirement Q_e of highest priority which requires attention by collapsing x and y into the same equivalence class. Then Q_e restrains the equivalence class of x (and y). This completes the construction at stage s. We show that the resulting ceer satisfies every Q_e. Suppose W_e is infinite. We claim that we eventually act for Q_e, hence Q_e will be satisfied. Each requirement of priority higher than Q_e acts at most once. Go to a stage t such that no requirement of higher priority acts after stage t. At most e many equivalence classes are restrained by requirements of higher priority. These classes cannot grow after stage t, so since W_e is infinite, there will eventually be distinct numbers $x, y \in W_e$ whose equivalence classes are not restrained by requirements of higher priority. Then we will act for Q_e (if we have not already done so). Finally, we show that there are infinitely many equivalence classes. In fact each class is finite: the sequence of collapses that forms a single class defines a sequence of requirements of higher and higher priority.

M1 ans. Every n-type is realized in a countable model which is an elementary extension of M. So, every n-type is realized in M. This is not yet enough to see that M is saturated, but this says that there are only countably many n-types for every n. Thus there is a countable saturated model. By universality for saturated models, it is an elementary extension of M. But then it is isomorphic to M, so M is saturated.

M2 ans. Let F be an infinite field. By the Ryll–Nardzewski theorem, it suffices to show that there are infinitely many n-types for some n. Suppose there is some non-algebraic element x. Then the pairs $(x, x), (x, x^2), (x, x^3), \ldots$ all have different 2-types. Thus we may assume that F has no non-algebraic element. But each algebraicity (i.e. a polynomial of degree k) has only finitely many realizations (at most k). Thus, we can choose a sequence of elements $x_1, x_2, x_3, \ldots$ no two of which are algebraic via the same polynomial. These have distinct 1-types.

M3 ans.

a) Take $\mathcal{N}$ to be a really big model of φ (upward Löwenheim–Skolem) and use Erdős–Rado to take an indiscernible subsequence. Since $\mathcal{N}$ modeled φ, so does every submodel. Since $\mathcal{L}$ is relational, every subset is a submodel. Alternatively, let T be a theory saying φ along with $(c_i)_{i\in\omega}$ is indiscernible (this is definable via a first order schema). Construct a model $\mathcal{N} \models T$ by compactness, using finite Ramsey in the given infinite model to find tuples satisfying finite parts of T. Let a_i be named by c_i in $\mathcal{N}$. Again, since $\mathcal{N}$ models φ, we can drop down to $\{a_i \mid i \in \omega\}$ modeling φ.

b) Going from finite to infinite: Restrict to the sublanguage $\mathcal{L}'$ generated by relations appearing in ψ. Suppose $a_1, \ldots a_n$ are elements in the given atomic-indiscernible model of $\varphi(x_1, \ldots, x_n)$. Let $p(x_1, \ldots, x_n)$ be the full atomic type of $(a_1, \ldots, a_n)$ in the language $\mathcal{L}'$. Extend $a_1, \ldots, a_n$ to an infinite sequence $\langle a_n \mid n \in \omega \rangle$ such that whenever $i_1 < \cdots < i_n$, then the $\mathcal{L}'$ atomic type of $a_{i_1}, \ldots, a_{i_n}$ is precisely p. For longer increasing sequences and relations in $\mathcal{L}\backslash\mathcal{L}'$, make arbitrary choices, preserving that $\langle a_n \mid n \in \omega \rangle$ is atomic-indiscernible. Atomic indiscernibility guarantees that $\langle a_n \mid n \in \omega \rangle \models \varphi$. In particular, this is an infinite model of φ. Going from infinite to finite: Use part (a) to get an infinite indiscernible sequence modeling φ. Universal sentences go down in substructures, so take $\mathcal{N}$ to be the first n points of the sequence.

c) Checking the $\mathcal{L}'$-structures of size n to see if any are atomic-indiscernible and satisfy φ is a finite process which checks whether or not φ has an infinite model, by (b).