Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let $L = \{0, 1, +\}$ where 0 and 1 are constant symbols, and $+$ is a binary function symbol. Let $M = (\mathbb{Z}, 0, 1, +)$, and let $T = \mathrm{Th}(M)$.

(a) Show that if $X \subseteq \mathbb{Z}$ is definable in M by a quantifier-free formula, then X is either finite or cofinite.

(b) Show that T does not have quantifier elimination by finding a definable subset of M that is neither finite nor cofinite.

E2. Let $L = \{+, -, 0\}$ be the language of abelian groups, where $-$ is the unary negation, i.e., the inverse operator. Let $Z = (\mathbb{Z}, +, -, 0)$. Show that there is an abelian group $G \equiv Z$ such that there exists a group embedding from $Z^\omega = (\mathbb{Z}^\omega, +, -, 0)$ into G. (Here $\mathbb{Z}^\omega$ is the collection of infinite sequences of integers with finite support, so Z^ω is the free abelian group on countably many generators.)

E3. Suppose that γ is an ordinal and $(U_\alpha)_{\alpha<\gamma}$ is a sequence of open subsets of $\mathbb{R}$ such that if $\alpha < \beta$, then $U_\alpha \subsetneq U_\beta$. Prove that γ is countable.

Set Theory

S1. Let R be a binary relation on the set A and let S be a binary relation on the set B. Assume that there is some forcing poset P that forces (A, R) to be isomorphic to (B, S). Then prove that the same holds for the specific poset $\mathrm{Fn}(\omega, \kappa)$, where $\kappa = \max(|A|, |B|)$.

S2. Prove that the following are equivalent in any model (M, E) of ZF:

1. (M, E) is a model of ZFC+V=HOD.
2. In (M, E), there is a definable well-ordering of the universe M (specifically, a definable total order of M isomorphic to the ordinals of (M, E)).
3. In (M, E), every definable nonempty set has an ordinal-definable element.

S3. Call a cardinal κ *worldly* if V_κ satisfies ZF. Show that the least worldly cardinal (if it exists) has cofinality ω.

Model Theory

M1. Let $\mathcal{A}$ be an infinite saturated model and let $\bar{a}$ be a tuple in $\mathcal{A}$. Prove that there is a strict substructure $B \subsetneq A$ and an isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$ so that $\pi(a) = a$ for each $a \in \bar{a}$.

M2.

- Let $T = \mathrm{Th}(\mathbb{R}, <, \mathbb{Z})$ be the theory of the real numbers with the usual ordering and a unary predicate for the integers. Is T $\aleph_0$-categorical?
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M3. Give an example of a theory T which is model complete but does not have quantifier elimination. Prove that T has these properties.

Computability Theory

C1. Show that if I is a nonempty index set that contains no index of any finite c.e. set, then I computes $0''$. (Recall that I is an *index set* if whenever $\varphi_e = \varphi_i$, then $e \in I$ if and only if $i \in I$.)

C2. Show that there are 1-generic sets $G, H \leq_T 0'$ such that $G \oplus H \equiv_T 0'$. (Give a direct construction; don't just quote the Posner–Robinson Theorem, which would be overkill.)

C3. An infinite set A is *effectively immune* if there is a computable function f such that for all e, if $W_e \subseteq$A then $|W_e| \leq f(e)$. Show that if a c.e. set B computes an effectively immune set A, then $B \geq_T 0'$.

Sketchy Answers or Hints

E1 ans. (a) Atomic formulas in at most one variable are (essentially) of the form $x = n$, or they are simply true or false. So such formulas only define singletons, everything, or nothing. Boolean combinations of such sets are either finite or cofinite.

(b) $(\exists y)x = y + y$ is true of x if and only if it is even.

E2 ans. We will use compactness. Let $L' = L \cup \{c_i\}_{i\in\omega}$. We intend for the new constants to freely generate a copy of Z^ω. To that end, let S be all sentences of the form:

$$a_0c_0 + a_1c_1 + \cdots + a_nc_n \neq 0,$$

where $n \in \omega$, $a_i \in \mathbb{Z}$ for all $i \leq n$, and at least one $a_i \neq 0$. (Here we are using shorthand for sentences that can be expressed in L'.) We want to show that $S \cup \mathrm{Th}(Z)$ is consistent; we will show that it is finitely satisfiable. Let S' be a finite subset of S. Let m be greater than any constant that appears in any sentence from S'. If we interpret c_i as m^i for each i, then $Z \models S'$, completing the proof. (This last claim follows from the fact that

$$(m-1) + (m-1)m + (m-1)m^2 + \cdots + (m-1)m^n = m^{n+1} - 1.)$$

E3 ans. Fix a countable basis $\{B_i\}_{i\in\omega}$ of open subsets of $\mathbb{R}$. (For example, the collection of open intervals with rational endpoints.) For each $\alpha < \gamma$, fix i_α such that $B_{i_\alpha} \subseteq U_{\alpha+1}$ but $B_{i_\alpha} \nsubseteq U_\alpha$. Note that the i_α's are distinct and that there are only countably many.

S1 ans. Consider a forcing extension $V[G]$, where G is generic for $\mathrm{Fn}(\omega, \kappa)$. Then A and B are countable. It's still the case that P that forces (A, R) to be isomorphic to (B, S); this follows from the product lemma (for product forcing). Consider an infinitely branching tree T coding all attempts to build an isomorphism between (A, R) and (B, S) using a back-and-forth argument. This tree is well-founded if and only if (A, R) and (B, S) are *not* isomorphic.

So assume for a contradiction that (A, R) and (B, S) are not isomorphic in $V[G]$. Then there is an ordinal labeling of T witnessing that it is well-founded. After forcing by P, this ordinal labeling still witness that T is well-founded, so (A, R) and (B, S) are not isomorphic in the forcing extension. But this is a contradiction. (Note that we have just given the proof of Π^1_1 absoluteness; you could quote Shoenfield's absoluteness theorem instead.)

S2 ans. $(1 \to 2)$: The usual HOD order is a definable well-ordering of the universe. $(2 \to 3)$: Select the least element with respect to the definable order. $(3 \to 1)$: If M thinks that there is a non-OD set, then the set of all non-OD sets in M of minimal rank is a definable nonempty set in M with no ordinal-definable elements.

S3 ans. Suppose κ is worldly. Fix an enumeration $(\varphi_i(x, y_1, \ldots, y_n))_{i\in\omega}$ of formulas in the language of set theory. (To simplify the verification, ensure that each formula is repeated infinitely often.) Given $\lambda < \kappa$ and $i \in \omega$, let $f_i(\lambda)$ be the least ordinal $> \lambda$ such that for every $x \in V_\lambda$, if there are $y_1, \ldots, y_n \in V_\kappa$ with $V_\kappa \models \varphi_i(x, y_1, \ldots, y_n)$, then there are such $y_1, \ldots, y_n$ in $V_{f_i(\lambda)}$. Replacement inside V_κ tells us that $f_i(\lambda) < \kappa$ whenever $\lambda < \kappa$. Now let $\lambda_0 = \omega$ and $\lambda_{i+1} = f_i(\lambda_i)$. By the Tarski–Vaught test, the limit of the sequence of λ_i's is worldly and clearly has cofinality ω. It is also at most κ by the previous sentence, so since κ is the least worldly cardinal, we're done.

M1 ans. Start by adding constants to the language to name $\bar{a}$. Now any elementary map fixes $\bar{a}$. Build a proper elementary superstructure $\mathcal{C}$ of $\mathcal{A}$ which has the same cardinality. Now use the universality of the saturated structure $\mathcal{A}$ to embed $\varphi : \mathcal{C}$ into $\mathcal{A}$. Then let $\mathcal{B} = \varphi(\mathcal{A})$.

M2 ans.

- No: Another model of this theory is obtained by having two copies of $\mathbb{Z}$, one left of the other, and then densifying.
- Yes: To show that any two countable models are isomorphic, first use a back-and-forth argument to get an isomorphism of the $\mathbb{Q}$-submodels. Then again use the back-and-forth method to extend this to a full isomorphism. The fact that both $\mathbb{Q}$, and $M \setminus \mathbb{Q}$, are dense within each other ensures this can be done.

M3 ans. Let M be the structure in the language of graphs consisting of a single 4-cycle. Let T be the theory of M.

C1 ans. We show that Tot can be reduced to I. Fix some $i_0 \in I$. By the s-m-n theorem, there is a computable (injective) function f such that

$$\varphi_{f(e)}(n) = \begin{cases} \varphi_{i_0}(n) & \text{if } (\forall m \leq n)\ \varphi_e(m)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then f witnesses that Tot $\leq_1 I$.

C2 ans. Construct $G = \bigcup_s \sigma_s$ and $H = \bigcup_s \tau_s$ simultaneously using an initial segment construction. Start with $\sigma_0 = \tau_0 = \emptyset$. At an even stage $s = 2e$, check whether there exists $\sigma \succeq \sigma_s$ such that $\sigma \in W_e$. If so, fix the least such σ, otherwise let $\sigma = \emptyset$. Define $\sigma_{s+1} = \sigma_s \hat{}\, \sigma \hat{}\, 0'(s)$ and $\tau_{s+1} = \tau_s \hat{}\, 0^{|\sigma|} \hat{}\, 1$. At odd stages switch the roles of τ_s and σ_s. Note that the construction is computable from $0'$. On the other hand $G \oplus H$ can determine what happened at each state of the construction and find the positions in each set that code $0'$.

C3 ans. Fix $A = \Gamma^B$ and let A be effectively immune via f. Use the recursion theorem with parameters to build for each k a c.e. set $W_{h(k)}$ such that $W_{h(k)}$ is initially empty but if k enters $0'$ at stage s then $W_{h(k)}$ is $(\Gamma^B \upharpoonright q)[t]$ for some q and $t \geq s$ such that $(\Gamma^B \upharpoonright q)[t]$ has $f(h(k)) + 1$ many members.

This means that if $k \in 0'$ at stage s, then $W_{h(k)}$ has more elements than $f(h(k))$ and hence can't be a subset of A. Thus B must change on the use of $(\Gamma^B \upharpoonright q)$ after stage s. So to compute $0'(k)$, wait for a stage s such that there is some q such that $(\Gamma^B \upharpoonright q)[s]$ has $f(h(k)) + 1$ elements and B does not change any more on the use of this computation. Then $0'(k) = 0'(k)[s]$.