**Instructions: Do all six problems.**

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

**E1.** Recall that a model $M$ is atomic if every tuple $\bar{a} \in M$ satisfies an isolated type. That is, there is some formula $\varphi$ so that $M \models \varphi(\bar{a})$ and for every formula $\rho(\bar{x})$, either $M \models \forall \bar{x}\, (\varphi(\bar{x}) \to \rho(\bar{x}))$ or $M \models \forall \bar{x}\, (\varphi(\bar{x}) \to \neg\rho(\bar{x}))$.

- Does there exist a countable theory with an atomic model of size $\aleph_0$ but no atomic model of size $\aleph_1$?
- Does there exist a countable theory with an atomic model of size $\aleph_1$ but no atomic model of size $\aleph_0$?

**E2.** Show that there is a partial recursive unary function which cannot be extended to a total recursive function.

**E3.** Let $\varphi$ be the Goldbach conjecture: Any even number $\geq 4$ is the sum of two primes. Let $T$ be a system of axioms extending ZFC, and suppose that $T \vdash \varphi$. Prove that $\mathrm{ZFC} + \mathrm{CON}(T) \vdash \varphi$.

Computability Theory

**C1.** Recall that an infinite $X \subseteq \omega$ is *immune* if it contains no infinite c.e. subset. A c.e. set is *simple* if its complement is immune. Let $A \subseteq \omega$ be a simple set.

1. [3.5 points] Show that there is a partial computable function $\varphi$ such that if $W_e$ is infinite, then $\varphi(e)\downarrow \in A \cap W_e$.
2. [6.5 points] Prove that the function in part (1) cannot be total.

**C2.** Let $A \geq_T 0'$. Construct sets $G$ and $H$ such that $G' \equiv_T A \equiv_T H'$ and $G$ and $H$ form a minimal pair.

**C3.** A $\Pi^0_1$ class $P \subseteq 2^\omega$ is *small* if for every computable function $g$ there is an $n$ such that fewer than $n$ strings in $2^{g(n)}$ are extendable in $P$. Use a priority argument to construct a nonempty, small $\Pi^0_1$ class with no computable elements. (A moveable markers construction is also perfectly acceptable.)

Model Theory

**M1.** Give an example of a theory with a countable prime model but no countable saturated model.

**M2.** Assume that $T$ is a theory in a language $\mathcal{L}$ with countably infinite signature.

- Show that if $T$ is $\omega$-categorical, then for every finite sublanguage $\mathcal{L}' \subseteq \mathcal{L}$, $T \upharpoonright \mathcal{L}'$ is $\omega$-categorical.
- Is the converse true? Prove or give a counterexample.

**M3.** Let $T$ be the theory of triangle-free symmetric graphs (i.e. the theory of symmetric graphs along with the axiom $\neg\exists x \exists y \exists z E(x,y) \wedge E(y,z) \wedge E(x,z))$ along with the additional axioms:

$$\begin{aligned}\Psi_{n,m} :=& \forall x_0, \dots x_{n-1} \forall y_0, \dots y_{m-1} \\ & \Bigg( \Bigg( \bigwedge_{j<k<m} \neg E(y_j, y_k) \wedge \bigwedge_{i<n, j<m} x_i \neq y_j \Bigg) \\ & \to \exists z \Bigg( \bigwedge_{i<n} \neg E(z, x_i) \wedge \bigwedge_{j<m} E(z, y_j) \Bigg) \Bigg)\end{aligned} \tag{1}$$

- Show that $T$ has a model.
- Show that $T$ has quantifier-elimination
- Show that $T$ is complete.

Set Theory

**S1.** Suppose $T \subseteq 2^{<\omega_1}$ is a subtree, i.e., $s \subseteq t \in T$ implies $s \in T$. Define $s =^* t$ iff they have the same domain and there are at most finitely many $\beta$ with $s(\beta) \neq t(\beta)$. Define

$$T^* = \{s \in 2^{<\omega_1} \ : \ \exists t \in T \ \ s =^* t\}.$$

Prove that if $T$ is an Aronszajn tree, then $T^*$ is also an Aronszajn tree. (Recall that an *Aronszajn tree* is an uncountable tree with no uncountable branches and no uncountable levels)

**S2.** Suppose $V = L$. Prove that for every $\alpha < \omega_1$ there exists $\delta < \omega_1$ such that

$$(L_{\delta+\alpha} \setminus L_\delta) \cap \mathcal{P}(\omega) = \emptyset.$$

**S3.** Let $\kappa$ be an uncountable singular cardinal. Let

$$\mathbb{P} = \{p : D \to 2 : D \in [\kappa]^{<\kappa}\}.$$

Prove forcing with $\mathbb{P}$ collapses $\kappa$ to $\mathrm{cof}(\kappa)$.

Sketchy Answers or Hints

**E1 ans.** Part 1: Yes, PA for example. Or the theory of equality and countably many distinct constants. The only atomic model is the one where every element is named by a constant. Part 2: No. Take an atomic model of size $\aleph_1$ and apply downward Skolem.

**E2 ans.** Let $\varphi(x) = \begin{cases} 1+\varphi_x(x) & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$ This is a partial computable function (e.g., making use of the universal Turing machine).

**E3 ans.** This can be done in 2 natural ways. The first uses the fact that ZFC (or even PA) "knows" that it proves every true $\Sigma^0_1$ sentence about arithmetic. This in turn is because it proves every true quantifier free sentence and you can build up from that to proving the correctness of a witness to the $\Sigma^0_1$ sentence. So, if the Goldbach conjecture were false, then its negation is a true $\Sigma^0_1$ sentence. So, ZFC knows that it would prove that. It is, ZFC proves $\neg\varphi \to \mathrm{PR}(\neg\varphi)$. But we also have $T \vdash \varphi$. So, ZFC proves that $\neg(\varphi)$ implies $\neg\mathrm{Con}(T)$, as needed.

A more set-theoretic approach: $\mathrm{Con}(T)$ means there's a model $M$ of $T$. $M$ has a (probably) nonstandard model of $\omega$ on which the Goldbach conjecture holds. But ZFC proves that $\omega$ is an initial segment of every nonstandard model of $\omega$ and Golbach is $\Pi_1$, so it holds on $\omega$.

**C1 ans.**

1. Fix uniformly computable approximations $\{A_s\}_{s<\omega}$ to $A$ and $\{W_{e,s}\}_{s<\omega}$ to $W_e$ for every $e$. Define $\varphi_s(e)$ to be the least element in $A_s \cap W_{e,s}$ if such exists and let it be undefined otherwise.

2. Assume that there is a total computable function $f$ with this property. Define a computable functions $g$ such that $W_{g(e)} = \overline{\{f(e)\}}$. By the fixed point theorem there is some $e$ such that $W_e = \overline{\{f(e)\}}$, but this contradicts the assumptions about $f$.

**C2 ans.** We build $G = \bigcup \sigma_s$ and $H = \bigcup \lambda_s$. At stage 0 we have $\sigma_0 = \lambda_0 = \emptyset$. Suppose that we have constructed $\sigma_s$ and $\lambda_s$. First we code $A$: let $\sigma^* = \sigma_s^\frown A(s)$ and $\lambda^* = \lambda_s^\frown A(s)$. Next we force the jump: Let $\Phi$ be the $s$-th Turing operator in some standard enumeration. For $\delta \in \{\sigma, \lambda\}$ ask if there is some extension $\eta \succeq \delta^*$ such that $\Phi^\eta(s)\downarrow$. If the answer is positive let $\delta^{**} = \eta$ for the least such $\eta$ and otherwise let $\delta^{**} = \delta^*$. Finally, we ensure the minimal pair requirements: ask if there are extensions $\tau \succeq \sigma^{**}$ and $\mu \succeq \lambda^{**}$, as well as some number $n$ such that $\Phi^\tau(n)\downarrow \neq \Phi^\mu(n)$. If the answer is positive then fix the least such pair of extensions $\tau$ and $\mu$ and let $\sigma_{s+1} = \tau$ and $\lambda_{s+1} = \mu$. Otherwise $\sigma_{s+1} = \sigma^{**}$ and $\lambda_{s+1} = \lambda^{**}$. Notice that the construction can be run by $A$, as $A \geq 0'$ and all questions have $0'$ computable answers. The construction can also be run by either $G \oplus 0' \leq_T G'$ or by $H \oplus 0' \leq_T H'$: use $G$ or $H$ to determine the bit $G(|\sigma_s|) = \sigma_{s+1}(|\sigma_s|) = A(s) = \lambda_{s+1}(|\lambda_s|) = H(\lambda_s)$ and $0'$ to answer the next two questions.

**C3 ans.** We build a $\Pi^0_1$ tree $T$ as follows. Let $Q_e$ be the requirement expressing that if $\varphi_e$ is a total $\{0,1\}$-valued function then it is not a branch $P$ and let $S_e$ be the requirement expressing that if $\varphi_e$ is a total function then there is an $n$ such that fewer than $n$ strings in $2^{g(n)}$ are extendable in $P$. At stage 0, we let $T = 2^{<\omega}$ and we assign a level $l_e = 2e$ to every $Q_e$ requirement and a witness $n_e = 2^{2e+1}$ to every $S_e$ requirement (the number of strings of length $l_e + 1$ in the current approximation to the tree). A $Q_e$ requirement requires attention at stage $s$ if $\varphi_{e,s}(l_e)\downarrow$. An $S_e$ requires attention at stage $s$ if $\varphi_{e,s}(n_e)\downarrow$. At stage $s$ pick the least requirement that requires attention and has not yet been satisfied. If there is no such requirement, let $T_{s+1} = T_s$. If this is $Q_e$ and $\varphi_e(l_e) \neq 0$ then let $T_{s+1} = T_s \setminus \{\sigma^\frown 1^\frown \tau \,|\, |\sigma| = l_e\}$. Injure all lower priority requirements by resetting their parameters and declare them unsatisfied; if $j = e + k$, where $k > 0$ then let $n_j = l_e + 2k$; if $j = e + k$, where $k \geq 0$ then let $n_j$ be the number of strings in $T_{s+1}$ of length $l_j + 1$.

If the requirement is $S_e$ then denote by $g(n)$ the number $\varphi_{e,s}(n_e)$. If $g(n) \leq l_e + 1$ then the requirement is already satisfied and we can let $T_{s+1} = T_s$. Otherwise let $T_{s+1} = T_s \setminus \{\sigma^\frown \tau^\frown \rho \,|\, |\sigma| = l_e + 1 \;\&\; |\tau| = g(n) - l_e - 1 \;\&\; \tau \notin \{0\}^{<\omega}\}$. Let $P = [\bigcap_s T_s]$.

**M1 ans.** Refining equivalence relations can work. Or (maybe more naturally), you have to come up with a theory where the isolated types are dense, yet there are uncountably many types (for some $n$). First construct a tree $T \subseteq 2^{<\omega}$ where the isolated paths are dense yet there are continuum many paths and then embed this as $S_1(T)$ for a theory $T$.

**M2 ans.** Use the Ryll-Nardzewski theorem and count types. If there were infinitely many inequivalent $\mathcal{L}'$-formulas $T \upharpoonright \mathcal{L}'$, then this would be infinitely many inequivalent $\mathcal{L}$-formulas in $T$. This gives a hint why the converse fails. Take the theory in language $\{U_i \mid i \in \omega\}$ with each $U_i$ unary that says that every possibility of $U$'s and negations occurs infinitely often. In every finite language of size $n$, you have exactly $2^n$ 1-types, and the theory is $\omega$-categorical, but in $T$, you have $2^{\aleph_0}$-many 1-types and the theory is not $\omega$-categorical.

**M3 ans.** Construct a model by successively adding elements to satisfy the various axioms $\psi_{n,m}$ without creating any triangles. For example, construct a model with universe $\omega$ by satisfying requirements $R_{\bar{x},\bar{y}}$ for tuples $\bar{x}, \bar{y} \subset \omega$ and order them so the each requirement appears infinitely often. When you visit a requirement, if the atomic type of the tuple $\bar{x}, \bar{y}$ hasn't been determined yet, just skip it to be considered later. If it has, and the antecedant of the axiom $\Psi_{|\bar{x}|,|\bar{y}|}$ holds, then add an element $z$ (the next number in $\omega$) so that $\bigwedge_{x\in\bar{x}} \neg R(z,x) \wedge \bigwedge_{y\in\bar{y}} R(z,y)$. Observe that if you do this by only connecting the new $z$ to elements in $\bar{y}$, then this cannot create a triangle.

To see that $T$ has QE: You could do this syntactically by eliminating an existential quantifier over a quantifier-free formula. The axioms $\Psi_{n,m}$ are exactly what you need to say that anything that doesn't make a triangle has to exist. Or you can use a semantic test: Given $\bar{a} \in M \models T$ and $\bar{b} \in N \models T$ so that $\bar{a} \cong \bar{b}$ and some element $c \in M$, let $n$ be the number of $a \in \bar{a}$ so that $E(a,c)$. Then $\Psi_{n,|\bar{a}|-n}$ implies that there is an element $d \in N$ so that $\bar{a}, c \cong \bar{b}, d$.

To see that $T$ is complete, you can just observe that there are no quantifier-free sentences in the language because there are no constants.

**S1 ans.** Suppose for contradiction that $b \in 2^{\omega_1}$ is a branch of $T^*$. For each countable ordinal $\alpha$ choose $t_\alpha \in T$ with $b{\upharpoonright}\alpha =^* t_\alpha$ and take $F_\alpha \subseteq \alpha$ finite such that $b\restriction(\alpha \setminus F_\alpha) = t_\alpha\restriction(\alpha \setminus F_\alpha)$. By the pushing down lemma there exists $\alpha_0 < \omega_1$ and a stationary set $S \subseteq \omega_1$ such that $F_\alpha \subseteq \alpha_0$ for all $\alpha \in S$. Splitting $S$ into countably many sets gives us a stationary subset $S_0 \subseteq S$ and $F$ with $F_\alpha = F$ for all $\alpha \in S_0$. Similarly there is a stationary set $S_1 \subseteq S_0$ and finite function $t$ with $t_\alpha\restriction F_0 = t$ for all $\alpha \in S_1$. But this means that $t_\alpha$ for $\alpha \in S_1$ is an $\omega_1$ branch of $T$.

**S2 ans.** Take $M$ countable elementary substructure of $L_{\omega_2}$ containing $\alpha$ and collapse it to $L_\beta$. Let $\delta = \omega_1^{L_\beta}$. Then since $L_{\omega_2}$ models that $\mathcal{P}(\omega) \subseteq L_{\omega_1}$ it follows that $L_\beta$ models that $\mathcal{P}(\omega) \subseteq L_\delta$. Since $\alpha \subseteq M$ the result follows.

**S3 ans.** Since $\mathbb{P}$ is cof($\kappa$)-closed no cardinals $\leq$ cof($\kappa$) are collapsed. Working in the ground model $M$ let $\kappa_i < \kappa$ for $i <$ cof($\kappa$) be a cofinal sequence of regular cardinals. For each $i$ let $h_i : \kappa_i \to \kappa_i$ be onto and $\kappa_i$-to-one.
Let $G$ be $\mathbb{P}$ generic over $M$ and put

$$X = \{\alpha : \exists p \in G \;\; p(\alpha) = 1\}.$$

Let $f : \mathrm{cof}(\kappa) \to \kappa$ be defined as follows. If $X \cap \kappa_i$ has a greatest element $\beta$ put $f(i) = h_i(\beta)$, otherwise put $f(i) = 0$. We claim that $f$ is onto. For contradiction suppose $\alpha < \kappa$ is not in the range of $f$ and $p \in G$ is such that

$$p \Vdash \forall i < \mathrm{cof}(\kappa) \;\; f(i) \neq \alpha.$$

Choose $i$ so that $|p| < \kappa_i$ and $\alpha < \kappa_i$. Since $h_i$ is $\kappa_i$-to-one and onto we can find $\beta < \kappa_i$ with $\kappa_i \cap \mathrm{dom}(p) \subseteq \beta$ and $h_i(\beta) = \alpha$. Extend $p$ to $q$ so that $q(\beta) = 1$ and $q(\gamma) = 0$ for all $\gamma$ with $\beta < \gamma < \kappa_i$. But then

$$q \Vdash f(i) = \alpha$$

which is a contradiction.