Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Prove that the theory of (ω, S) where $S(n) = n + 1$ is not finitely axiomatizable.

E2.

1. Let $T_0, \ldots, T_n$ be $\mathcal{L}$-theories such that each $\mathcal{L}$-structure is a model of exactly one T_i. Show that each T_i is finitely axiomatizable.
2. Show that this may fail for an infinite collection of theories $T_0, T_1, T_2, \ldots$ that partition the collection of $\mathcal{L}$-structures.

E3. Show that the cardinality of the continuum, $\mathfrak{c} = 2^{\aleph_0}$, does not have countable cofinality. Give a direct proof; do not simply quote a theorem.

Model Theory

M1. Say that $\mathcal{M}$ is minimal if it has no proper elementary submodels.

1. Give an example of a theory with a prime model that is not minimal.
2. Show that if a complete theory T has a prime model and a minimal model, then they are isomorphic.
3. Show that $\mathrm{Th}(\mathbb{Z}, +)$ has a minimal model that is not prime.

M2. Show that there is no completion T of the partial theory of fields in the language $\{+, \cdot, 0, 1\}$ which is $\aleph_0$-categorical.

M3. Show that a theory with quantifier elimination has an axiomatization by sentences of the form $\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y})$ where φ is quantifier-free.

Computability Theory

C1. Consider the function

$$f(e) = \begin{cases} n & \text{if } n \text{ is the least number not in } W_e, \\ -1 & \text{if } W_e = \omega. \end{cases}$$

Show that f is not majorized by a $\emptyset'$-computable function.

C2. Assume that $A \subseteq \omega$ is noncomputable. Construct a set B such that $B \not\geq_T A$ but $B' \geq_T A$.

C3. Assume that X is a noncomputable c.e. set. Show that there is a simple set A that does not compute X. Recall that A is *simple* if it is a coinfinite c.e. set and it intersects nontrivially every infinite c.e. set (i.e., A is a c.e. set whose complement is *immune*).

Sketchy Answers or Hints

E1 ans. The theory is axiomatizable as follows: (1) Different elements have different successors. (2) There is a unique element that is not a successor. (3) For each $n \geq 1$: No element is its own nth successor. (To see that these axioms give a compete theory, note that they have a unique model of size $\aleph_1$.) Now if the theory has a finite axiomatization, then a finite subset F of the axioms above is sufficient—only what is needed to prove the axioms in the finite axiomatization. To finish, note that since F only has finitely many axioms of type (3), it cannot rule out a sufficiently big loop.

E2 ans. (1) Since $T_i \cup T_j$ is not satisfiable when $i \neq j$, there is a finite subset $B_{i,j}$ that is not satisfiable. Let $T_i^j = B_{i,j} \cap T_i$. Then we claim that $S_i = \bigcup_{j \neq i} T_i^j$ axiomatizes T_i. It is enough to show that it is satisfied by the same models. Since $S_i \subseteq T_i$, it holds on any model of T_i. On the other hand, if $\mathcal{M}$ is not a model of T_i, then it is a model of T_j for some j. But then $\mathcal{M}$ does not satisfy $T_i^j \subseteq S_i$. (2) In the empty language, let T_0 be the theory saying that there are infinitely many distinct elements, and for each $i > 0$, let T_i be the theory saying that there are exactly i elements. These partition all structures, but T_0 is not finitely axiomatizable (by the same sort of argument used in E1).

E3 ans. (This is a special case of König's theorem.) Let $\{r_\alpha\}_{\alpha<\mathfrak{c}}$ list all elements of 2^ω. Let $\lambda_0 < \lambda_1 < \cdots$ be an ω-sequence of ordinals limiting to $\mathfrak{c}$. We build a sequence $r \in 2^\omega$ by columns as follows (i.e., we are using the fact that $2^\omega \approx 2^{\omega\times\omega}$). Make the nth column of r different from the nth column of r_α for all $\alpha < \lambda_n$. This is possible because $|\lambda_n| < |2^\omega| = \mathfrak{c}$. But now note that r is different from every element of 2^ω, which is a contradiction.

M1 ans. 1. Consider the theory of an infinite set with no other structure. 2. If M is prime and N is minimal, then use primeness of M to show that it elementarily embeds into N. But minimality of N shows that this embedding is onto. So $M \cong N$. 3. The structure $(\mathbb{Z}, +)$ is itself minimal: Suppose M is an elementary submodel, and n is an integer in M. Then elementarity of M

shows that n is divisible in M by n, showing that $1 \in M$ and so all of $\mathbb{Z}$ is contained in M. To see it's not prime, use omitting types to get a model N so that every element of N is divisible by some natural number > 1. Then you cannot elementarily embed $\mathbb{Z}$ into N since 1 has nowhere to go.

M2 ans. In fields, there can be only finitely many elements satisfying a polynomial $p(x)$. In particular, for every n, there can be only finitely many elements x so that $x^n = 1$. It follows then by compactness that there is a countable model of T containing an element of infinite order. But then you have infinitely many 2-types in T corresponding to pairs (x, y) where $y = x^n$ and $y \neq x^m$ for any $m < n$.

M3 ans. We build a new theory T' as follows: We include the set of elimination sentences: $\forall \bar{x}(\exists \bar{y}\psi(\bar{x}, \bar{y}) \leftrightarrow \rho(\bar{x})$. These sentences explain how the QE is witnessed. Then add the 1-quantifier theory of T into T'. We then must argue that T' is an axiomatization of T. Since $T' \subseteq T$, we need only show every model of T' is a model of T. If $\varphi := \forall \bar{x}\rho(\bar{x}) \in T$, we use the QE axioms to replace ρ by a quantifier-free version ρ'. Both T and T' agree that φ is equivalent to $\forall \bar{x}\rho'(\bar{x})$, and then we have placed this formula into T', so any model of T' is a model of φ. Arguing similarly for formulas beginning with $\exists \bar{x}$, we see that any formula in T is modeled by any model of T'.

C1 ans. Assume, for a contradiction, that there is a $\emptyset'$-computable function g that majorizes f. Note that $W_e = \omega$ if and only if $[0, g(e)] \subseteq W_e$, so $\emptyset' \oplus g \equiv_T \emptyset'$ can compute TOT, which contradicts the fact that TOT$\equiv_T \emptyset''$.

C2 ans. This is very similar to the proof of the Friedberg Jump Inversion theorem. Build B by initial segments $\beta_0 \preceq \beta_1 \preceq \cdots$. On even stages, we ensure that $\varphi_e^B \neq A$ as follows: if there is an e-split, take the first discovered e-split of β_{2e} and take the useful side. (If no e-split exists, then φ_e^B is partial or computable.) On odd stages, let $\beta_{2e+2} = \beta_{2e+1}A(e)$. Note that $B' \geq_T B \oplus \emptyset' \geq_T A$ because $B \oplus \emptyset'$ can determine the sequence $\{\beta_i\}$. For even stages, $\emptyset'$ can determine when e-splits occur and using B you can see which side of the split was taken. Odd stages just require reading off the next bit of B, which is the next coded bit of A.

C3 ans. This should be proved using a finite injury argument. The requirements are $P_e\colon |W_e| = \infty \implies A \cap W_e \neq \emptyset$ and $N_e\colon \varphi^A \neq X$. The strategy for P_e is the same one used in the construction of a finite simple set. The *Sacks preservation* strategy should be used for N_e.