Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on a piece of paper and hand it to a proctor, who will contact a grader as soon as possible.

E1. Consider the following conditions on a collection of sets $\mathcal{A} \subseteq \mathcal{P}(\omega)$:

1. $\mathcal{A}$ contains all finite sets.
2. $\mathcal{A}$ is closed under complements, finite unions, and finite intersections.
3. If $X \in \mathcal{A}$ is infinite, then there are disjoint infinite sets $Y, Z \in \mathcal{A}$ such that $X = Y \cup Z$.

Show that if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega)$ are countable and satisfy conditions (a)–(c), then $(\mathcal{A}, \subseteq) \cong (\mathcal{B}, \subseteq)$.

E2. Let $T \subseteq 2^{<\omega}$ be a co-c.e. tree. Show that there is a c.e. theory D and a bijection π between the consistent completions of D and the infinite paths through T. Furthermore, if $\overline{D}$ is a completion of D, then it is Turing equivalent to the corresponding path $\pi(\overline{D})$.

(You may make D a propositional theory.)

E3. Let $S \subseteq \mathbb{Z}^2$. A tile is a set of the form $[a, b] \times \{c\}$ or $\{c\} \times [a, b]$ where $a \leq b$ and c are integers. A ray is a set of the form $[a, \infty) \times \{c\}$, $(-\infty, b] \times \{c\}$, $\{c\} \times [a, \infty)$, or $\{c\} \times (-\infty, b]$, where a, b, and c are integers. Suppose that for every n the set S is a disjoint union of tiles of size n. Does this entail that S is a disjoint union of rays?

Model Theory

M1. Let T be a theory and let $M \models T$ be a countable model with the following properties:

1. M is homogeneous.
2. Whenever $M \subseteq N \models T$ is countable, there exists an elementary extension $M' \succ N$ such that $M \cong M'$.

Show that M is a saturated model of T.

M2. Suppose that $M \models T$ and T is totally transcendental. Let S be a non-empty set of definable sets in M. Show that there is a definable set $X \in S$ so that whenever Y is a proper definable subset of X, either $Y \notin S$ or $X \setminus Y \notin S$.

M3. Let $L = \{<, U\}$ where U is a unary relation symbol and $<$ is a binary relation symbol. Let T_0 be the axioms saying that $<$ is a total order (note that T_0 does not mention U). Prove that there are exactly 9 complete L-theories containing T_0 which have quantifier elimination.

Computability Theory

C1.

1. (4 points) Show that every finite Boolean combination of c.e. sets is Turing reducible to $\emptyset'$ by a Turing functional with computably bounded use.

2. (6 points) Show that this may fail for some Δ^0_2-set A.

C2. Let $I \subseteq 2^\omega$ be a countable collection of sets that is closed downwards under Turing reducibility. Suppose X computes an enumeration $\{X_n\}_{n\in\omega}$ of I (i.e., $I = \{X_n : n \in \omega\}$, possibly with repetitions) so that the set $F = \{n : X_n \text{ is finite}\}$ is $\Delta_2(X)$. Show that X can compute an enumeration of the infinite sets in I.

C3. Build noncomputable sets $A, B \leq_T \emptyset'$ such that if G is 2-generic and Y is computable from both $A \oplus G$ and $B \oplus G$, then $Y \leq_T G$.

Hint: Recall that G is 2-generic if it meets or avoids every Σ^0_2 set of binary strings. In the verification, use the Σ^0_2 sets

$$U_{e,i} = \{\sigma \in 2^{<\omega} \colon (\exists n)\ \varphi_e^{A\oplus\sigma}(n)\downarrow \neq \varphi_i^{B\oplus\sigma}(n)\downarrow\}.$$

Build A and B so that these sets are useful.

Sketchy Answers or Hints

E1 ans. We prove this using a a back-and-forth construction. Denote by X^0 the set $\overline{X}$ and let $X^1 = X$. At stage $n+1$ of the construction we can assume that we have built finite sequences $A_0, \ldots A_{n-1}$ and $B_0, \ldots, B_{n-1}$ so that for every Boolean vector $\alpha \in 2^n$ we have that the cardinality of $\bigcap_{i<n} A_i^{\alpha(i)}$ equals the cardinality of $\bigcap_{i<n} B_i^{\alpha(i)}$. If n is even we go forth: Let A_n be the first set that has not yet appeared in our sequence. Now for each $\alpha \in 2^n$ we can find a set $B_n^\alpha \subseteq \bigcap_{i<n} B_i^{\alpha(i)}$ in $\mathcal{B}$ so that that $A_n \cap \bigcap_{i<n} A_i^{\alpha(i)}$ has the same cardinality as $B_n^\alpha \cap \bigcap_{i<n} B_i^{\alpha(i)}$ and $\overline{A_n} \cap \bigcap_{i<n} A_i^{\alpha(i)}$ has the same cardinality as $\overline{B_n^\alpha} \cap \bigcap_{i<n} B_i^{\alpha(i)}$. We take the union of all B_n^α to form B_n. Since $\{\bigcap_{i<n} A_i^{\alpha(i)} : \alpha \in 2^n\}$ partitions ω we can show that the new sequences $A_0, \ldots A_n$ and $B_0, \ldots B_n$ have the same property with respect to all $\alpha \in 2^{n+1}$. At odd stages we do the back direction.

E2 ans. Our language will have propositional variables $\{p_n\}_{n\in\omega}$. For $\sigma \in 2^{<\omega}$ of length n, let $\varphi_\sigma = p_0^{\sigma(0)} \wedge p_1^{\sigma(1)} \wedge \cdots p_{n-1}^{\sigma(n-1)}$, where $p_i^0 = \neg p_i$ and $p_i^1 = p_i$. Let $D = \{\neg\varphi_\sigma \colon \sigma \notin T\}$, which is a c.e. theory. Any consistent completion $\overline{D}$ of D determines the values of the propositional variables and is Turing equivalent to $X_{\overline{D}} = \{n \colon p_n \in \overline{D}\}$. By choice of D, $X_{\overline{D}}$ is a path through T. Conversely, any infinite path through T corresponds to a consistent assignment of truth values to the variables, hence a completion of D.

E3 ans. This can be done with propositional compactness, but I will use topological compactness. Let T_n be the tiling of S with tiles of size n. Let $X_n \in 2^\omega$ code T_n as follows:

$$X_n = \{\langle\langle a,b\rangle, \langle c,d\rangle\rangle \colon \langle a,b\rangle, \langle c,d\rangle \in S \text{ and they are in the same tile in } T_n\}.$$

By compactness, $\{X_n\}$ has a convergent subsequence, say converging to X. The following is the key point, and takes a little checking: as a limit of tilings of S with bigger and bigger tiles, X must code a tiling of S with rays and "double rays" (i.e., whole rows or columns). The double rays can be split into rays, giving us the desired tiling.

M1 ans. We first observe that every n-type consistent with T is realized in M: Given any p consistent with T, build a countable model A realizing p. Since $A \equiv M$, there is some N which is an elementary extension of both. And with downward Skolem, N can be chosen countable. Then condition 2 says there's an elementary extension of this N which is isomorphic to M. So, M realizes p. It follows from homogeneity and that all types are realized that M is saturated: Given a type $p(x, \bar{a})$ over a tuple $\bar{a}$ in M, we know that $M \models p(c, \bar{d})$ for some $c, \bar{d} \in M$. But then $tp(\bar{a}) = tp(\bar{d})$, so homogeneity gives some element e so that $tp(e, \bar{a}) = tp(c, \bar{d}) = p$. So, p is realized and M is saturated.

M2 ans. Let $A \in S$ (since S is non-empty). If A works as our X, yay. If not, take a $Y_0 \in S$ and $Y_1 \in S$ so that $Y_0 \sqcup Y_1 = A$. Repeat for these Y_i's. This process must end at some point because otherwise we build an infinite binary tree of formulas over M. Being totally transcendental means that this cannot happen.

M3 ans. Let T be a completion of T_0 with QE and suppose $M \models T$ Observe that every single element in U in M satisfies the same QF-type. So, if U is finite in A, then the first and last elements satisfy the QF-type, but not the same type, unless U has size 0 or 1. Thus U is either size 0, 1, or ∞. Same for $\neg U$. Next, observe that if U is infinite, then U is dense without endpoints. Again, being infinite, there are $x < y \in U$ with something between them. Thus every pair must have something in U between them. Similarly, there is an element which is not an endpoint, so they are all not an endpoint. Same for $\neg U$. Next, if U has size 1 and is $\{a\}$, then a must be the least or greatest element. This is because any $x, y \in \neg U$ must have the same type, so you can't have e.g., $x < a < y$. Next, consider the case where both U and $\neg U$ are infinite. If $x < y < z$ with $x, z \in U$ and $y \in \neg U$, then every pair $x, z \in U$ must be as such. Same with roles reversed. So, there are 3 cases: They are dense-co-dense, every element of U is less than every element of $\neg U$ or vice versa. Writing out the possibilities consistent with the above, you should find exactly 9:

- U is empty and $<$ is a DLOW
- $\neg U$ is empty and $<$ is a DLOW

- U is a singleton which is the least element. $\neg U$ is a DLOW
- U is a singleton which is the greatest element. $\neg U$ is a DLOW
- $\neg U$ is a singleton which is the least element. U is a DLOW
- $\neg U$ is a singleton which is the greatest element. U is a DLOW
- U and $\neg U$ are both DLOW's and they are dense co-dense.
- U and $\neg U$ are both DLOW's with $a \in U$, $b \notin U$ implying $a < b$.
- U and $\neg U$ are both DLOW's with $a \in \neg U$, $b \in U$ implying $a < b$.

C1 ans.

1. Every c.e. set is in fact m-reducible to the halting problem K, so a Boolean combination of c.e. sets requires only a fixed finite number of calls to K.
2. Build a Δ_2^0-set A such that $A(\langle e, i\rangle)$ diagonalizes against the reduction Φ_e with computable use bound φ_i.

C2 ans. Given $\{X_n\}_{n\in\omega}$, build a new list $\{Y_n\}_{n\in\omega}$ such that $Y_{\langle m,t\rangle}$ is computed as follows: Copy X_m as long as $m \notin F_s$ for $s \geq t$. If we see $m \in F_s$ for some $s \geq t$, then make $Y_{\langle m,t\rangle}$ cofinite by putting every remaining element into it. (Note that all cofinite sets are in I.) If $m \notin F$, then there is a large enough t such that $m \notin F_s$ for all $s \geq t$, so $Y_{\langle m,t\rangle} = X_m$.

C3 ans. We build sets $A = \bigcup_{s\in\omega} \alpha_s$ and $B = \bigcup_{s\in\omega} \beta_s$ using an initial segment construction. Start with $\alpha_0 = \beta_0 = \emptyset$. At even stages $s = 2e$ we ensure that A and B are not computable: for $\delta \in \{\alpha, \beta\}$ if $\varphi_e(|\delta_s|)\downarrow = 1$ then set $\delta_{s+1} = \delta_s 0$ and otherwise set $\delta_{s+1} = \delta_s 1$. At odd stages $s = 2(e, i, j) + 1$ let j code the finite binary string σ. If there are $\alpha \succeq \alpha_s$, $\beta \succeq \beta_s$, $\tau \succeq \sigma$, and natural number n such that $\varphi_e^{\alpha\oplus\tau}(n)\downarrow \neq \varphi_i^{\beta\oplus\tau}(n)\downarrow$ then let $\alpha_{s+1} = \alpha$ and $\beta_{s+1} = \beta$ for the least such α and β. Otherwise set $\alpha_{s+1} = \alpha_s$ and $\beta_{s+1} = \beta_s$. If G is 2-generic and $X = \varphi_e^{A\oplus G} = \varphi_i^{B\oplus\sigma}$ the G must avoid $U_{e,i}$, say by its initial segment σ. Let j be σ's code and let $s = 2(e, i, j) + 1$. Now X is computable as $X(n) = \varphi_e^{\alpha\oplus\tau}(n)$ for the least $\alpha \succeq \alpha_s, \beta \succeq \beta_s, \tau \succeq \sigma$ such that $\varphi_e^{\alpha\oplus\tau}(n)\downarrow$.