Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

1. Let $\{K_n : n \in \omega\}$ be a collection of nonempty pairwise disjoint finitely axiomatizable classes of L-structures. Show that there is an L-structure not in any K_n.

2. Call a total function f large if $f(e) > \varphi_e(0)$ whenever $\varphi_e(0)$ is defined. Show that if f is large, then f computes $0'$.

3. A *perfect tree on Cantor space* is a subset T of $2^{<\omega}$ which is closed under taking initial segments and so that every element of T has two incomparable (not necessarily immediate) extensions, i.e., T is a tree with no leaves and no isolated paths.

A perfect tree T on Cantor space is *pointed* if every path through T computes T.

- Show that if T is a pointed perfect tree on Cantor space and x computes T, then x is Turing equivalent to some path through T.
- Show that if T is a pointed perfect tree on Cantor space, then T has a uniformly pointed perfect subtree. That is, T contains a subset U which is also a perfect tree on Cantor space and there is a single Turing functional Φ so that whenever x is a path through U, then $U = \Phi^x$.

4. Directly prove (i.e., don't just quote a theorem) that there are disjoint low c.e. sets A and B such that $A \cup B = \emptyset'$.

5. Show

- The theory of $(\mathbb{R}, +, 0)$ has exactly two 1-types and exactly $\aleph_0$ 2-types.
- The theory of $(\mathbb{R}, +, <, 0)$ has exactly three 1-types and exactly $2^{\aleph_0}$ 2-types.

6. Let $\mathcal{M}$ be a countable $\aleph_0$-categorical structure. Show that if $X \subseteq \mathcal{M}^n$ is invariant under all automorphisms of $\mathcal{M}$, then X is definable.

Sketchy Answers or Hints

1 ans. Let K_n be axiomatized by φ_n for each n. Use compactness to show that $T = \{\neg\varphi_n\}_{n\in\omega}$ is consistent. In particular, the finite fragment $\{\neg\varphi_n\}_{n<m}$ is satisfied by every structure in the (nonempty) class K_m, hence is consistent. But if $\mathcal{M} \models T$, then it is not in any K_n.

2 ans. Using the smn theorem, uniformly find, for each $i \in \omega$, an index e_i so that $\varphi_{e_i}(0)$ is the least stage at which i enters the halting set K (or diverges if i never enters K). Then f computes $0'$ by checking, for each i whether or not $i \in K_{f(e_i)}$.

3 ans. For the first problem: Inductively define a path through T as follows: Start with $\tau_0 = \emptyset$. Extend τ_0 to be the first branching encountered. Extend to τ_1 by going left if $0 \in x$ and and right if $0 \notin x$. Then extend to the next branching and let τ_2 be the result of going left if $1 \in x$ and right if $1 \notin x$. Repeating defines a sequence y. Clearly y is a path through T, thus y computes T. Observe that $y \oplus T$ computes x by reading off where it branched left vs right. Thus y computes x. Similarly, x computes T and $x \oplus T$ computes y, so x computes y. For the second part: Force with a sequence of perfect subtrees of T. Try to find a subtree U_0 where you have ensured that any path x through U_0 has $\varphi_0^x \neq T$ (i.e., if you can force partiality by taking the tree above some point, do that. If you can't, then take the subtree where you force totality, and if someone gets something wrong, you take the tree above that node). If this is possible, observe that the U_0 you found is computable from T. Next find a subtree U_1 where you ensure every path x in U_1 has $\varphi_1^x \neq T$. Eventually you cannot, because otherwise you find a path through T which does not compute T. That means that your forcing gives you a perfect tree U_k so that every path has $\varphi_k^x = T$. But each $U_0, \ldots, U_k$ is computable from T, so composing φ_k with the computation $T \geq_T U_k$ gives your Φ.

4 ans. This is a finite injury argument. Fix a computable injective enumeration of $\emptyset'$, say $\{k_s\}_{s<\omega}$. For every pair e, x and $Y \in \{A, B\}$ we will have a lowness requirement $L^Y_{e,x}$ stating that if there are infinitely many stages s such that $\Phi^Y_{e,s}(x)\downarrow$ then $\Phi^Y_e(x)\downarrow$. Each lowness requirement imposes a restriction $M^Y_{e,x}$. We order the lowness requirements with order type ω. At stage s set $M^Y_{e,x}$ to be the length of the use if $\Phi^{Y_s}_{e,s}(x)\downarrow$. Let $L^Y_{e,x}$ be the highest priority requirement such that $k_s < M^Y_{e,x}[s]$. If $Y = A$ then enumerate k_s in B and if $Y = B$, enumerate k_s in A. An induction shows that every lowness requirement will be satisfied.

5 ans. For the first part: In $(\mathbb{R}, +, 0)$, any two elements which are non-zero are automorphic. Thus there is a single type p so that for every $\varphi \in p$, the theory contains the formula $\forall x(x \neq 0 \to \varphi(x))$. It follows that the theory has only two 1-types: That of 0 and that of a non-zero element. For the $\aleph_0$-many 2-types, consider the pairs $(1, n)$ for any $n \in omega$. For the second part, run the same trick with the 3 1-types of 0, positive elements, and negative elements. For the continuum many 2-types, for each r, let p_r be the type of the pair $(1, r)$. Show that these are different for each real r by using a rational between any given r_1 and r_2. Alternatively, you could directly show QE for these two theories and extract the result from QE.

6 ans. There cannot be two tuples $\bar{x} \in X$ and $\bar{y} \notin X$ of the same type, since, by saturation of $\mathcal{M}$, there would be an automorphism moving $\bar{x}$ to $\bar{y}$. So, X is a union of $p(M)$ for some types p. But by $\aleph_0$-categoricity, every type is isolated, so defined by a first-order formula, and there are only finitely many of them, so X is definable.