Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

1. Let T be the theory of the structure $(\mathbb{N}, S, 0)$ where S is a unary function symbol interpreted as the function $S(x) = x + 1$.

1. Show that the theory T has quantifier elimination.
2. Give an explicit axiomatization of T and show that your axiomatization implies all of T.

2. Let A be a collection of c.e. sets which is downward closed under Turing reducibility. We say that A is represented by a set of natural numbers R if $A = \{W_i \mid i \in R\}$. Show that if A is represented by a Σ_3 set, then it is represented by a computable set.

3. Build a continuum size set $\mathcal{A}$ of Turing degrees that is independent, i.e., no member of $\mathcal{A}$ is Turing reducible to the join of any finite collection of other degrees in $\mathcal{A}$.

4. Let $\mathcal{M}$ be an elementary substructure of $\mathcal{N}$ and let $A \subset M$ be a subset. Show that $acl_{\mathcal{M}}(A) = acl_{\mathcal{N}}(A)$. Here, $acl_{\mathcal{C}}(A)$ denotes the algebraic closure of A taken in the structure $\mathcal{C}$ (i.e., $b \in acl_{\mathcal{C}}(A)$ if there is a formula $\varphi(x) \in \mathcal{L}_A$ so that $\{y \in C \mid \mathcal{C} \models \varphi(y)\}$ is finite and contains b). Give an example showing that this is not true if you only assume $\mathcal{M}$ is a substructure of $\mathcal{N}$ even under the assumption that $\mathcal{M} \equiv \mathcal{N}$.

5. Let T be a complete consistent countable theory. Show that there is a countable model $\mathcal{M}$ of T so that for any tuple $\bar{a}$ in either the type of $\bar{a}$ is isolated or there exists a formula $\varphi(\bar{x})$ so that $\mathcal{M} \models \varphi(\bar{a})$ and there are no isolated $|\bar{a}|$-types containing $\varphi(\bar{x})$.

6. Let $U \subseteq 2^\omega$ be a dense G_δ set (recall a G_δ set is a countable intersection of open sets). Build a $G \in 2^\omega$ and a dense G_δ set $V \subseteq 2^\omega$ such that if $H \in V$, then $G \oplus H \in U$.

Sketchy Answers or Hints

1 ans.

1. Use the criterion given, e.g., in Marker (Cor. 3.6): Let $\mathcal{A}$ be a common substructure of $\mathcal{M}, \mathcal{N} \models T$, and fix a quantifier-free formula $\psi(\overline{v}, w)$ and $\overline{a} \in A$ such that there is $b \in M$ with $\mathcal{M} \models \psi(\overline{a}, b)$. There are now two cases: If ψ includes an equation between b and some $a \in A \cup \{0\}$, then $b \in A$ and so $b \in N$ since $\mathcal{A}$ is a substructure. Otherwise, ψ does not relate b to any $a \in A \cup \{0\}$ by an equation, so let k be a bound on the number of times the S-symbol is used in ψ and simply let $c \in A \subseteq N$ be more than k many S-steps away from any $a \in A \cup \{0\}$; so $\mathcal{N} \models \psi(\overline{a}, c)$.

2. T is axiomatized by saying that S is a bijection from M onto $M \setminus \{0\}$ and that for all x and all $n > 0$, $S^n(x) \neq x$. This is all we used in part 1 to prove quantifier elimination, and this axiomatization clearly implies all quantifier-free facts about a model of T.

2 ans. Since A is downward closed under Turing reducibility, it contains in particular all finite sets. Now represent $i \in R$ as $\exists j\, (|W_{f(i,j)}| = \infty)$. We now define a computable representation of A by $\{W_i \upharpoonright \max(W_{f(i,j)}) \mid i, j \in \omega\}$, where the max of an infinite c.e. sets is ∞.

3 ans. See Soare (1987), Exercise VI.1.8. This exercise only builds pairwise incomparable degrees, but can be easily modified to get our version by applying to the Padding Lemma so that each reduction is defeated infinitely often, and by letting the reduction use the join of all but one oracle at a level of the F-tree.

4 ans. If $\mathcal{M} \preceq \mathcal{N}$, then an algebraic formula $\varphi(x)$ (with parameters from A) has the same number of solutions in $\mathcal{M}$ and in $\mathcal{N}$. Since clearly $acl_{\mathcal{M}}(A) \subseteq acl_{\mathcal{N}}(A)$, we obtain the first part.

A possible counterexample for the second part is gotten by letting $\mathcal{M}$ be the positive integers, and $\mathcal{N}$ the non-negative integers, both in the language of successor.

5 ans. If there are no isolated types for any tuple from $\mathcal{M}$, then it is easy to build a full binary tree of consistent formulas such that $\varphi_{\sigma 0}$ and $\varphi_{\sigma 1}$ are pairwise inconsistent and for each $i < 2$, $\varphi_{\sigma i}$ implies φ_{σ}.

6 ans. Let $U = \bigcap_{i\in\omega} U_i$, where each U_i is a dense open set, and let $\{\sigma_j\}_{j\in\omega}$ be an enumeration of $2^{<\omega}$. We build G by initial segments $g_0 \preceq g_1 \preceq g_2 \preceq g_3 \cdots$, where g_0 is the empty string. We also build dense open sets V_i for each $i \in \omega$; these start out empty. *Construction. Stage* $s = \langle i, j\rangle$. (We ensure that $G \oplus H \in U$.) Let $[\tau_0 \oplus \tau_1] \subseteq U_i$ be such that $\tau_0 \succ g_s$ and $\tau_1 \succeq \sigma_j$. Let $g_{s+1} = \tau_0$ and put $[\tau_1]$ into V_i. *Verification.* Let $G = \bigcup_{i\in\omega} g_i$ and $V = \bigcap_{i\in\omega} V_i$. Note that we ensured that each V_i is a dense open set and that if $H \in V_i$, then $G \oplus H \in U_i$.