Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

1. Fix $\mathcal{L}$ a first order language and T a universal $\mathcal{L}$-theory. Suppose that T is model complete. Let $\mathcal{A} = (A, \dots)$ be a model of T and $f : A \to A$ be definable in $\mathcal{A}$ without parameters. Show that there are finitely many $\mathcal{L}$-terms $t_1(x), \dots t_n(x)$ each with at most 1 variable so that for every $a \in A$, $f(a) \in \{t_1(a), \dots, t_n(a)\}$.

2. For each of the following, provide either a proof or a counterexample.

(a) If a complete countable first-order theory T has a countable prime model, must it have a countable saturated model?

(b) If a complete countable first-order theory T has a countable saturated model, must it have a countable prime model?

(c) If a complete countable first-order theory T has a countable saturated model which is also prime, must T be $\aleph_0$-categorical?

3. Call a function $f\colon \omega \to \omega$ *infinitely often equal* if for every computable $g\colon \omega \to \omega$ we have $(\exists^\infty n)\, f(n) = g(n)$. Prove that the following are equivalent for a Turing degree $\mathbf{d}$:

1. $\mathbf{d}$ is hyperimmune (i.e., there is a $\mathbf{d}$-computable function that is not dominated by any computable function.
2. $\mathbf{d}$ computes an infinitely often equal function.

Note: $(1 \Rightarrow 2)$ is worth 9/10 points.

4. Let $\mathcal{L}$ be the signature $\{<, f\}$ where $<$ is a binary relation symbol and f is a unary function symbol. Let T be the theory saying that $<$ is a dense linear order and that f is an order-preserving bijection such that $f(x) > x$ for all x.

(a) Show that T has quantifier elimination.

(b) Show that T is complete.

(c) Is T $\aleph_0$-categorical?

5. Prove that there is a non-computable c.e. set $X \subseteq \omega$ such that for every computable $f\colon \omega \to \omega$ there exists an e such that $X \upharpoonright f(e) = \varphi_e \upharpoonright f(e)$.

6. Assuming that PA is consistent, show that there is a sentence φ in the language of arithmetic that is independent of PA and such that neither $\mathsf{PA} + \varphi \vdash \mathrm{Con}(\mathsf{PA})$, nor $\mathsf{PA} + \neg\varphi \vdash \mathrm{Con}(\mathsf{PA})$.

Sketchy Answers or Hints

1 ans. Suppose first that there was some $a \in A$ so that $f(a)$ is not equal to a term applied to a. Then we could take the substructure of $\mathcal{A}$ comprised of the terms over a. By the universality of T, this is also a model of T. By the model completeness of T, this is an elementary substruture of $\mathcal{A}$. But it does something differently for $f(a)$, contradicting the elementarity. Thus $f(a)$ is always a term in a. Now, suppose that there were not finitely many terms so that $f(a)$ was always one of these for each $a \in A$. Then run a compactness argument to build an elementary extension $\mathcal{B}$ of $\mathcal{A}$ containing an element a so that $f(a)$ is not a term in a and get the same contradiction above.

2 ans. No, e.g., arithmetic, Yes, and yes. If a theory only has countably many n-types, then the isolated ones are dense by a bit of topology – complete perfect spaces are uncountable. If the saturated model is prime, then all the types are isolated. Apply Ryll-Nardzewski.

3 ans. If f is dominated by g, then it is not infinitely often equal to $g + 1$, so $(\neg 1 \Rightarrow \neg 2)$ is immediate. For $(1 \Rightarrow 2)$, let h be a $\mathbf{d}$-computable function that is not dominated by any computable function. We build an infinitely often equal function f. Let $f(\langle e, k\rangle) = \varphi_{e,h(k)}(\langle e, k\rangle)$ if the right side converges. Otherwise, let $f(\langle e, k\rangle) = 0$. For verification, assume that φ_e is total. Because h is not dominated by the modulus function of $k \mapsto \varphi_e(\langle e, k\rangle)$, there will be infinitely many k such that $f(\langle e, k\rangle) = \varphi_e(\langle e, k\rangle)$.

4 ans. We can use the QE-test. We are given $\bar{a} \in A$ and $\bar{b} \in B$ with $\langle \bar{a}\rangle_A \cong \langle \bar{b}\rangle_B$ and an element $c \in A$. We need to find a $d \in B' \succeq B$ so that $\langle \bar{a}c\rangle_A \cong \langle \bar{b}d\rangle_B$. By the order-preserving-ness and bijectivity of f, we just have to find a d in the right cut amongst the $\{f, f^{-1}\}$-closure of $\bar{b}$. By the density of $<$, this defines a consistent type, so it's realized in some $B' \succeq B$. Completeness follows immediately from QE since T has no constant symbols. T is not $\aleph_0$-categorical since there are infinitely many 2-types. In particular, for each n there is a 2-type saying that $y > f^{(n)}(x)$ but $y \leq f^{(n+1)}(x)$.

5 ans. We build X using a priority tree. Odd level nodes of length $2e+1$ ensure that $X \neq \overline{W}_e$ by the usual strategy (pick a fresh witness, wait for it to enter W_e with one outcome, if does, move to a different outcome and enumerate the witness in X). A node α of length $2i$ builds a computable function $\varphi_{e(\alpha)}$ where by the recursion theorem we may assume that we know $e(\alpha)$. It keeps $\varphi_{e(\alpha)}$ undefined until (if ever) $\varphi_i(e(\alpha)) = u$ is defined. While waiting it has one outcome. If the awaited event happens it makes $\varphi_{e(\alpha)} \upharpoonright u = X \upharpoonright u$ and switches outcomes (effectively imposing a restraint on $X \upharpoonright u$).

6 ans. First solution: Consider the Rosser sentence ρ that states that for every proof of ρ from PA there is a proof of $\neg\rho$ with smaller Gödel index. It is straightforward to check that if PA is consistent then $\mathsf{PA} \not\vdash \rho$ and $\mathsf{PA} \not\vdash \neg\rho$. In other words, $\mathrm{Con}(\mathsf{PA}) \vdash \mathrm{Con}(\mathsf{PA}+\rho)\ \&\ \mathrm{Con}(\mathsf{PA}+\neg\rho)$. Thus, by the fact that no theory T extending PA can prove its own consistency, we have the desired result.

Second solution: Let us replace $\mathrm{Con}(\mathsf{PA})$ by any sentence ψ that is not provable from PA (implicitly assuming that PA is consistent). Since $\mathsf{PA}+\neg\psi$ is consistent and every consistent, c.e. extension of PA is incomplete (the Gödel–Rosser theorem), we can fix a φ that is independent of $\mathsf{PA}+\neg\psi$ (hence independent of PA). If $\mathsf{PA}+\varphi \vdash \psi$, then $\mathsf{PA}+\neg\psi \vdash \neg\varphi$, which we have just assumed to be false. Therefore, $\mathsf{PA}+\varphi \nvdash \psi$. By the same argument, $\mathsf{PA}+\neg\varphi \nvdash \psi$.

Third solution: Again, replace $\mathrm{Con}(\mathsf{PA})$ by any sentence ψ that is not provable from PA. Assume, for a contradiction, that if φ is independent of PA, then either $\mathsf{PA}+\varphi \vdash \psi$ or $\mathsf{PA}+\neg\varphi \vdash \psi$. In other words, for each φ at least one of the following holds:

- $\mathsf{PA} \vdash \varphi$, in which case $\mathsf{PA}+\varphi$ is consistent.
- $\mathsf{PA} \vdash \neg\varphi$, in which case $\mathsf{PA}+\neg\varphi$ is consistent.
- $\mathsf{PA}+\varphi \vdash \psi$, in which case $\mathsf{PA}+\neg\varphi \nvdash \psi$, or else we would have $\mathsf{PA} \vdash \psi$. Hence $\mathsf{PA}+\neg\varphi$ is consistent.
- $\mathsf{PA}+\neg\varphi \vdash \psi$, so as above, $\mathsf{PA}+\varphi$ is consistent.

This means that for each sentence φ, we can computably pick one of φ or $\neg\varphi$ that is guaranteed to be consistent with PA. Such power wold allow us to compute a separator of the computably inseparable c.e. sets $\{e\colon \varphi_e(e) = 0\}$ and $\{e\colon \varphi_e(e) = 1\}$, hence we have a contradiction.