Instructions: Do all six problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 (US Letter) sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

1. We say that $A \leq_{wtt} B$ if $A = \Phi^B$ for some Turing functional Φ so that the use of the computation is bounded by a total computable function. Show that if A, B, and C are three c.e. sets such that $A \leq_{wtt} B \oplus C$ then A is the disjoint union of c.e. sets B_0 and C_0 such that $B_0 \leq_{wtt} B$ and $C_0 \leq_{wtt} C$.

2. Suppose θ is a sentence that is independent from PA. Show that the set $\{\varphi : \varphi \text{ is a sentence and } PA + \varphi \vdash \theta\}$ is not computable.

3. A set of natural numbers A is density-1 if the limit of the densities of its initial segments is 1, or in other words, if

$$\lim_{n\to\infty} \frac{|A \cap \{0, 1, \dots n-1\}|}{n} = 1.$$

1. A set A is *generically computable* if there exists a partial computable function φ whose domain is density-1 such that $\varphi(n) = \chi_A(n)$ for all $n \in \text{dom}(\varphi)$. Show that if A is 1-generic then A is not generically computable.

2. A set A is *coarsely computable* if there exists a total computable $\{0, 1\}$-valued function χ so that the set of all x so that $\chi(x) = \chi_A(x)$ has density-1. Show that if A is 1-generic then A is not coarsely computable.

4. Let $\mathcal{L}_n$ be the signature $\{<, R_0, \ldots, R_n\}$ where $<$ is binary and each R_i is unary. Let T_n be the $\mathcal{L}_n$-theory which says that $<$ is a dense linear order without endpoints, the relations R_i define disjoint sets, every element satisfies one of the relations R_i, and each R_i holds on a dense subset (i.e., if $x < y$, then there exists a z so that $x < z \wedge z < y \wedge R_i(z)$).

Let $\mathcal{L}_\infty$ be the signature $\{<\} \cup \{R_i \mid i \in \omega\}$, and let T_∞ be the theory which says that $<$ is a dense linear order without endpoints, the relations R_i define disjoint sets, and each R_i holds on a dense subset.

(a) Show that T_n is $\aleph_0$-categorical.

(b) Show that T_n has Quantifier Elimination.

(c) Is T_∞ $\aleph_0$-categorical?

(d) Does T_∞ have Quantifier Elimination?

5. Suppose that $\mathcal{M}$ is a countable saturated structure, that $X \subseteq M$ is definable (with parameters), and suppose that X is fixed as a set by every automorphism of $\mathcal{M}$, i.e., if $\sigma : M \to M$ is an automorphism of $\mathcal{M}$, then $x \in X \leftrightarrow \sigma(x) \in X$ for every $x \in M$.

Show that X is definable without parameters in $\mathcal{M}$. (Hint: Consider the set of tuples that could be used in place of the chosen parameters – if this set has a definable subset, you may replace the given parameters by reference to members of this set.)

6. Show that replacing saturation by homogeneous-ness in the previous problem does not suffice. That is, there exists a $\mathcal{M}$ which is homogeneous with a definable (with parameters) set X which is fixed as a set by every automorphism of $\mathcal{M}$, yet X is not definable without parameters.

Sketchy Answers or Hints

1 ans. Fix $A \leq_{wtt} B \oplus C$ and let Φ and f witness this. We need to build B_0 and C_0 so that $A = B_0 \cup C_0$ and wtt-reductions Γ from C_0 to C and Λ from B_0 to B. Fix c.e. approximations to A, B, and C. At stage s we consider the length of agreement l_s between $A[s]$ and $\Phi^{B\oplus C}[s]$. If $x < l_s$, $A_s(x) = 0$, and $\Gamma^B(x) \uparrow$ and $\Lambda^C(x) \uparrow$ then we set both of these to be 0 with use $f(x)$. If $x \in A_s \setminus B_0 \cup C_0[s]$ then we must pick one of C_0 or B_0 to enumerate x in. Since $x < l_s$ it must be that at least one of $\Gamma^B(x) \uparrow$ or $\Lambda^C(x) \uparrow$. Indeed, if both are defined and equal to 0 then this definition was done at a previous stage t based on a computation of $\Phi^{B\oplus C}(x)[t] = 0$, which is not valid any longer, so either $B_t \upharpoonright f(x) \not\preceq B_s$ or $C_t \upharpoonright f(x) \not\preceq C_s$. Suppose $B_t \upharpoonright f(x) \not\preceq B_s$. Then we, enumerate x in B_0 and update the computation $\Gamma^B(x)[s] = 1$ with use $B_s \upharpoonright f(x)$. Otherwise, we do a similar action with C_0 and Λ. Finally, if $A_s(x) = 1$ and $x \in (B_0 \cup C_0)[s]$ then we update the operators Γ and Λ if either is undefined using use up to $f(x)$. Note, the reason this works is that we always use the bound $f(x)$ when defining a computation of x in Γ or Λ and so once both oracles settle on their initial segments up to $f(x)$, the computation will stop changing. Further, if $A(x)$ changes from 0 to 1 we know that a change must have appeared below the bound $f(x)$. If this were Turing reducibility what could happen is $\Phi^{B\oplus C}(x) = 0$ with use u_1, so we define Γ and Λ accordingly, then $B \upharpoonright u_1$ changes and a new computation $\Phi^{B\oplus C}(x) = 0$ appears with use $u_2 > u_1$. And then x enters A accompanied by a change in C above u_1 but below u_2. This means that $\Phi^{B\oplus C}(x) = 1$ is correct again, but we cannot correct $\Gamma^B(x)$ or $\Lambda^C(x)$.

2 ans. Let $A = \{\varphi : \varphi \text{ is a sentence and } PA + \varphi \vdash \theta\}$. Suppose A were computable. Let R be a computable axiomatization of PA. List the sentences in $\overline{A}$ as $\tau_1, \ldots \tau_n, \ldots$. We build $T = R \cup \bigcup_{n<\omega} T_n$ as follows: We start with $T_0 = \emptyset$. $T_{n+1} = T_n \cup \{\tau_n\}$ if $\tau_n \wedge \bigwedge_{\tau \in T_n} \tau \notin A$ and $T_{n+1} = T_n$ otherwise. Note that T is computable. By induction, we show that for all n, we have $R \cup T_n \nvdash \theta$ and so (by compactness) it follows that T is consistent. Furthermore, T is a theory that extends PA. Indeed, if $T \vdash \sigma$ then $T, \sigma \nvdash \theta$ and so if n is such that $\tau_n = \sigma$, we will have that $\sigma \in T_{n+1}$, because $\tau_n \wedge \bigwedge_{\tau \in T_n} \tau \notin A$. Thus T is a computable consistent extension of PA, which is not possible.

3 ans.

1. Note that A is generically computable if and only if there are c.e. sets C_0 and C_1 so that $C_0 \subseteq \overline{A}$, $C_1 \subseteq A$ and $C_0 \cup C_1$ has density 1. Indeed suppose that A is generically computable and let φ witness this. The set $C_0 = \{x : \varphi(x) = 0\}$ and the set $C_1 = \{x : \varphi(x) = 1\}$ are c.e. sets that satisfy the claim. On the other hand, given c.e. sets C_0 and C_1, then we can define a partial computable function φ with domain $C_0 \cup C_1$ so that $C_0 = \{x : \varphi(x) = 0\}$ and the set $C_1 = \{x : \varphi(x) = 1\}$. Fix a 1-generic set A. If $C_1 \subseteq A$ is c.e. then C is finite. Indeed, consider the set $S = \{\sigma : \exists n(\sigma(n) = 0 \wedge C_1(n) = 1\}$. If $C \subseteq A$ then A must avoid S and the only way that this can be is if C_1 is finite. Similarly if $C_0 \subseteq \overline{A}$ then C_0 is finite (because $\overline{A}$ is 1-generic). It follows that $C_0 \cup C_1$ is a finite set and cannot be density 1.

2. Suppose that A were 1-generic and fix a total computabel χ. Then for every n consider the c.e. set S_n consisting of all σ such that $|\sigma| \geq n$ and $\frac{|\{x:\sigma(x)=\chi(x)\}|}{|\sigma|} < \frac{1}{2}$. The set S_n is dense: indeed given and σ, we can extend it by a string of length $|\sigma|+1$ so that the resulting string τ differs from χ on all bits $x \geq |\sigma|$. A must meet S_n for every n, so the sequence $\frac{|\{x:A(x)=\chi(x)\}\cap\{0,1,\ldots n-1\}|}{n}$ infinitely often dips below $\frac{1}{2}$, it can't have limit 1.

4 ans. a) Back and forth. The density of each R_i allows you to carry out the back-and-forth. b) Let $\bar{a} \in \mathcal{M} \models T_n$ and $\bar{b} \in \mathcal{N} \models T_n$ be isomorphic tuples and let $c \in \mathcal{M}$ be another element. The same argument as in the back-and-forth in (a) shows that there exists an element $d \in \mathcal{N}$ (no need to go to an elementary extension) so that $\bar{a}c \cong \bar{b}d$. Thus, this theory satisfies a condition for QE. c) T_∞ cannot be $\aleph_0$-categorical, since there are infinitely many different 1-types separated by the predicates R_i. d) T_∞ does have QE. Again, use the QE test. The case that is different is where the element c is not in any of the R_i's. Then you might have to pass to an elementary extension to find d (an application of compactness shows that the partial type of an element in a given interval which is not in any of the R_i's is consistent).

5 ans. Let X be defined by $\varphi(x, \bar{a})$ where $\bar{a}$ is some tuple of parameters used to define X. Then for any $\sigma \in \mathrm{Aut}(\mathcal{M})$ and $b \in M$, $b \in X \leftrightarrow \sigma(b) \in X$. In particular, $\varphi(b, \bar{a}) \leftrightarrow \varphi(\sigma(b), \bar{a})$. But this is the same as saying $\varphi(b, \bar{a}) \leftrightarrow$

$\varphi(b, \sigma^{-1}(\bar{a}))$. Thus, for any $\sigma \in \mathrm{Aut}(\mathcal{M})$, $\sigma(\bar{a})$ would be as good a parameter as $\bar{a}$ for defining X via φ. But a tuple $\bar{a}'$ is an automorphic image of $\bar{a}$ if and only if it satisfies the same type. Let $p(\bar{y})$ be the type of $\bar{a}$. Then the following is inconsistent (otherwise it would be realized in $\mathcal{M}$, by saturation): $\{\exists x(\varphi(x, \bar{a}) \not\leftrightarrow \varphi(x, \bar{y}))\} \cup p(\bar{y})$. By compactness, there is a single formula $\theta(\bar{y}) \in p(\bar{y})$ so that $\{\exists x(\varphi(x, \bar{a}) \not\leftrightarrow \varphi(x, \bar{y})), \theta(\bar{y})\}$ is inconsistent. Then X is defined by $\exists \bar{y}\theta(\bar{y}) \wedge \varphi(x, \bar{y})$, which requires no parameters.

6 ans. Many examples can be found. Here is one: Let $\mathcal{L}$ be the signature $\{c_i \mid i \in \omega\}$ of countably many constants. Let $\mathcal{M}$ be the structure where each constant names a distinct element and there is exactly 1 element not named by a constant. Check that 1) $\mathcal{M}$ is homogeneous. In fact, every tuple satisfies a distinct type from every other tuple, so the hypothesis of homogeneousness is trivial. 2) if X is the set that is the single element not named by a constant, then X is preserved by all automorphisms of $\mathcal{M}$ (there's only 1) and X is definable with a parameter ($x = a$), but is not definable without a parameter. You can show that X is not definable without a parameter in a couple ways. You could: a) Show the $\mathcal{L}$-theory just saying that each c_i names a distinct element is complete and has QE – then use QE to see what is definable without parameters. OR b) Suppose it was defined as $\varphi(x)$ where φ mentions only the constants c_i with $i < K$. Then in the reduct of $\mathcal{M}$ to the language comprising only these constants, this same formula φ still defines 1 element. But in this reduct, any element not named by a constant is automorphic to any other. So, φ cannot hold for only one of them.