Logic SEP: Day 1

July 15, 2013

1 Some references

- Syllabus:
  http://www.math.wisc.edu/graduate/guide-qe
- Previous years' qualifying exams:
  http://www.math.wisc.edu/~miller/old/qual/index.html
- Miller's Moore style notes:
  http://www.math.wisc.edu/~miller/old/m771-98/logintro.pdf
- Kunen's notes:
  http://www.math.wisc.edu/~kunen/770.html
- Shoenfield's Mathematical Logic
- Cohen's Set theory and the continuum hypothesis, Chapter 1

2 First order logic

2.1 Syntax and semantics

Suppose T is theory and ϕ is a sentence in some first order language L. We write $T \vdash \phi$ if ϕ can be formally deduced from T; i.e. there is a proof of ϕ in which the unjustified assumptions are members of T. We'll never go into the details of the exact definition of a formal proof and assume that the reader has seen at least one such definition. The only important thing to remember is that the notion of a formal proof is effective in that one can write a computer program to check if a sequence of sentences forms a formal proof.

A first order structure $\mathcal{M}$ has a universe, which we'll denote by M, and names for some members and finitary relations and functions over this universe. The signature of $\mathcal{M}$ is the set of all names for constants, relations and functions and these determine the non logical vocabulary of a first order language L. We sometimes say that $\mathcal{M}$ is an L-structure.

A formula $\phi(\vec{x})$ is true/valid in a structure $\mathcal{M}$ at a tuple $\vec{a} \in M$, and we write $\mathcal{M} \vDash \phi(\vec{a})$ if $\phi(\vec{a})$ is actually true in $\mathcal{M}$ when its non logical symbols are interpreted in the appropriate way. If T is a set of sentences (formulas with no free variables), we write $\mathcal{M} \vDash T$, read $\mathcal{M}$ models T, if every sentence in T is true in $\mathcal{M}$. We write $T \vDash \phi$ if every model of T is also a model of ϕ. In the case when $T = 0$, we write $\vDash \phi$ and such sentences ϕ are called (logically) valid. The complete theory of a structure $\mathcal{M}$, $Th(\mathcal{M})$, is the set of all sentences true in $\mathcal{M}$.

The completeness theorem says that $T \vDash \phi$ is same as $T \vdash \phi$ and therefore this is a finitary notion. A more informative version of this theorem says that if T is consistent; i.e., not all sentences can be formally derived from T, then T has a model with a universe of size not more than $max(|T|, \omega)$ where $|T|$ is the size of T. A much more interesting theorem is the following:

Theorem 1 (Compactness theorem) *A first order theory T has a model iff every finite subset of T has a model.*

2.2 Lowenheim-Skolem-Tarski theorems

Let $\mathcal{M}$ and $\mathcal{N}$ be two first order structures with same signature. $\mathcal{M}$ is isomorphic to $\mathcal{N}$, written $\mathcal{M} \cong \mathcal{N}$ if there is a bijection between M and N that preserves the interpretation of their signature. We say that $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, written $\mathcal{M} \equiv \mathcal{N}$ if they have the same theories; i.e. a sentence is true in one iff it is true in the other. We say that $\mathcal{M}$ is a substructure of $\mathcal{N}$, and write $\mathcal{M} \subseteq \mathcal{N}$ if $M \subseteq N$ and the constants, relations, functions in $\mathcal{M}$ are obtained by restricting the corresponding things in $\mathcal{N}$. If, in addition to $\mathcal{M} \subseteq \mathcal{N}$, every formula with parameters from M is true in $\mathcal{M}$ iff it's true in $\mathcal{N}$, we say that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ and write $\mathcal{M} \preceq \mathcal{N}$. Clearly, each one of $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{M} \preceq \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$. The following is a basic tool to test if a given structure is an elementary substructure of another structure:

Theorem 2 (Tarski-Vaught criterion) *Let $\mathcal{M} \subseteq \mathcal{N}$. Suppose for each formula $\phi(x, \vec{c})$ with x free and with parameters $\vec{c}$ from M, whenever there is some $b \in N$ such that $\mathcal{N} \vDash \phi(b, \vec{c})$ there is also some $a \in M$ such that $\mathcal{N} \vDash \phi(a, \vec{c})$. Then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.*

One can use the last criterion to readily establish the following:

Theorem 3 (Downward LST) *Let $\mathcal{M}$ be an infinite first order structure over a language L. Let $A \subseteq M$ and suppose $max(\omega, |L|, |A|) \leq \kappa \leq |M|$. Then there is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ of size κ whose universe contains A.*

The next theorem follows easily from the downward LST and the compactness theorem.

Theorem 4 (Upward LST) *Let $\mathcal{M}$ be an infinite first order structure over a language L. Let $\kappa \geq max(|M|, |L|, \omega)$. Then there is an elementary superstructure $\mathcal{N}$ of $\mathcal{M}$ of size κ.*

2.3 Definability and automorphisms

Let $\mathcal{M}$ be a structure. An n-ary relation R on M is definable (without parameters) in $\mathcal{M}$ if there is a formula $\phi(\vec{x})$ such that $R = \{\vec{a}|\mathcal{M} \vDash \phi(\vec{a})\}$. It is definable with parameters in $\mathcal{M}$ if for some formula $\phi(\vec{x}, \vec{y})$ and elements $\vec{b} \in M$, $R = \{\vec{a}|\mathcal{M} \vDash \phi(\vec{a}, \vec{b})\}$. As an exercise, describe the definable (with and without parameters) subsets of the field of real numbers, $(\mathbb{R}, 0, 1, +, .)$. Definability of an n-ary function on M is same as the definability of corresponding $(n+1)$-ary relation on M. An automorphism of $\mathcal{M}$ is a bijection on M that preserves all constants, functions and relations in the signature of $\mathcal{M}$. The following is an easy but useful fact for establishing undefinability.

Theorem 5 *Definable objects in $\mathcal{M}$ are fixed under every automorphism of $\mathcal{M}$. If parameters are being used, then the previous statement holds for all automorphisms that fix the parameters.*

2.4 Undefinability of truth

For every sufficiently rich structure, it is impossible to code its truth predicate as a parameter free definable relation over the structure. We state the theorem for arithemtic $(\omega, 1, +, .)$. Let $\ulcorner\phi\urcorner$ denote the Godel number of a formula ϕ. Let $T = \{\ulcorner\phi\urcorner|(\omega, 1, +, .) \vDash \phi\}$.

Theorem 6 *There is no formula $T(x)$, which defines T in $(\omega, 1, +, .)$.*

2.5 Completeness, categoricity and axiomatizability

A theory is said to be κ-categorical if all of its models of size κ are isomorphic. It is complete if for each sentence ϕ it proves either ϕ or its negation $\neg\phi$. If a theory T is κ-categorical for some $\kappa \geq max(|T|, \omega)$ then it is complete.

For a language L, a class of L-structures $\mathbb{C}$ is said to be axiomatizable if there is an L-theory T such that $\mathbb{C} = \{\mathcal{M}|\mathcal{M} \vDash T\}$. An example of a non axiomatizable class is the class of all finite structures (over any language L). Proof: If T is any theory with arbitrarily large finite models then T has an infinite model by compactness theorem.

2.6 Quantifier elimination

A theory T admits quantifier elimination if every formula in the language of T is equivalent, relative to T, to a quantifier free formula. If a theory T decides every quantifier free sentence in its language and admits quantifier elimination then it is complete. For examples of the application of this idea see the proof of the completeness and decidability of Presburger arithemtic and the theory of real closed fields.

2.7 Back and forth constructions

Suppose $(L, <)$ is a countable dense linear order with no largest or least element. We want to show that there is essentially only one such linear order, viz. the rationals $(\mathbb{Q}, <)$. Enumerate $L = \{a_1, a_2, ...\}$ and $\mathbb{Q} = \{b_1, b_2, ...\}$. Start by sending a_1 to b_1. To decide where to send a_n, look at the position of a_n with respect to the points in L that have already been mapped into $\mathbb{Q}$ and send it to some b_m which has the same position with respect to the images of those points. To ensure that the limiting map is a bijection, take care of a_n and b_n by step $2n$. This construction of an isomorphism between countable dense linear orders without end points is called a back and forth construction. Such constructions can be used to establish the ω-categoricity of theories like atomless boolean algebras and random graphs.

3 Computable sets

3.1 Computably enumerable sets

A set is computably enumerable (c.e.) iff it is the range of some computable function iff it is the domain of some partial computable function iff it is the projection of some computable subset of ω^2. A set is computable iff both it and its complement are c.e. The halting problem is c.e. but not computable.

3.2 Decidability and computably axiomatizable theory

A countable theory T, is decidable if the set $\{\phi | T \vdash \phi\}$ is computable. T is computablty axiomatizable if it has a computable set of axioms. PA (Peano arithmetic) is an example of a computably axiomatizable undecidable theory.

Theorem 7 (Godel's first incompleteness theorem) *Every computably axiomatizable extension of PA is incomplete.*

4 Well orderings and transfinite recursion

A linear order $(L, <)$ is a well order iff every subset of L has a $<$-least element. An ordinal is the order type of a well order. For basic ordinal arithmetic see Kunen's book. We only mention transfinite induction here.

Theorem 8 (Transfinite recursion) *Let κ be a cardinal, s be a set and suppose G is a function from sets to sets. Then there is a unique function f with domain κ such that $f(0) = s$ and for every ordinal $\alpha < \kappa$, $f(\alpha) = G(f \restriction \alpha)$.*

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1 Examples on first order logic

- (Jan 2012) Prove that the set of validities in the language with two unary operation symbols is undecidable. You may assume without proof that the set of validities in the language with one binary relation symbol is undecidable.
- (Jan 2012) Suppose T_i for $i < n$ (with $n < \omega$) are L-theories such that every L-structure $\mathcal{M}$ satisfies exactly one of the T_i. Prove that each T_i is finitely axiomatizable If $n = \omega$ must this still be true? Prove or give a counterexample.
- (Aug 2011) Let $T = Th(\mathbb{Z}, +)$. Prove that T has uncountably many pairwise nonisomorphic countable models
- (Jan 2011) Let $L = \{E\}$, where E is a binary relation symbol. Let Σ in L be the axioms that say that E is an equivalence relation. Let ϕ be a sentence of L which is consistent with Σ. Prove that ϕ is true in some finite model of Σ.
- (Jan 2011) Let L be the language with two non-logical symbols f and g. Let T declare that f and g are bijective functions which commute with each other, as well as the axiom scheme stating for all m and n integers (including negative ones) which are not both zero that $\forall x(f^m(g^n(x)) = x)$. Show that T is complete, decidable, and not finitely axiomatizable.
- (Aug 2010) Let L be the language containing one binary relation symbol. A graph is a symmetric irreflexive binary relation. It is n-colorable iff there is a map from its universe into n such that no two elements in the relation are assigned the same value.
(a) Show that there is a first order L-theory T whose models are exactly the 3-colorable graphs.
(b) Prove that T is not finitely axiomatizable.
- (Jan 2010) The class of simple groups (groups with no non trivial normal subgroup) is not axiomatizable.

- (Aug 2009) Let B be a disjoint union of compact intervals with rational end points. Let $A = B \cap \mathbb{Q}$. Then $\mathcal{A} = (A, <)$ is an elementary substructure of $\mathcal{B} = (B, <)$.

- (Jan 2009) For any set A, show that there is a collection $\mathcal{C}$ of subsets of A such that (a) A finite subset of A is in $\mathcal{C}$ iff it has even size and (b) Whenever X, Y are disjoint subsets of A, $X \cup Y \in \mathcal{C}$ iff either $X, Y \in \mathcal{C}$ or $X, Y \notin \mathcal{C}$.

  Hint: Use compactness theorem.

- (Aug 2008) Let A, B be disjoint infinite sets of integers and let $\mathcal{M} = (M, R)$ be the structure with universe $M = A \cup B \cup (A \times B)$ and ternary relation $R = \{(a, b, (a, b)) | a \in A, b \in B\}$. Show that $T = Th(\mathcal{M})$ doesn't have a finite set of axioms.

  Hint: Write an infinite list of axioms that axiomatizes T and show that no finite fragment is enough.

- (Aug 2008) Let T be a consistent axiomatizable theory with only finitely many complete extensions in the same language. Show that T is decidable. (Here, a theory is a set of sentences closed under deduction, and it is axiomatizable if it is the deductive closure of a computable set of sentences.)

- (Jan 2007) PA denotes Peano Arithemtic. Let $\mathcal{M} \models PA$ and let $pr(M) = \{p \in M | \mathcal{M} \models p \text{ is prime }\}$. Show the following:
  (a) $|M| = |pr(M)|$.
  (b) For every $S \subseteq pr(M)$, there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that there is some $a \in M$ for which $S = \{p \in pr(M) | \ p \text{ divides } a \ \}$.
  (c) Every complete extension of PA has continuum many models.

- (Aug 2006) A Σ_2 formula has the form $\exists\vec{x}\forall\vec{y}\phi$ where ϕ is quantifier free. A Σ_3 formula has the form $\exists\vec{x}\forall\vec{y}\exists\vec{z}\phi$ with ϕ quantifier free.
  (a) Show that for Σ_2 sentences ϕ, $(\omega, <) \models \phi$ iff $(\omega + \omega, <) \models \phi$.
  (b) Give an example of a Σ_3 sentence ϕ for which the above fails.

- (Jan 2006) Characterize the definable subsets of $(\omega, <)$.

- (Aug 2004) Show that every non standard model of true arithmetic, viz $\mathcal{A} = Th(\omega, <, 0, S, +, .)$, has a substructure which is not a model of $\mathcal{A}$.

- (Jan 2003) Let $\{U_n | n \in \omega\}$ be unary relation symbols. Let T be a theory that says that $U_0 \supseteq U_1 \supseteq U_2 \supseteq ...$ and that each one of $U_n \backslash U_{(n+1)}$ as well as complement of U_0 is infinite. Show that T is complete.

  Consider the ω-categorical theories T_k = set of sentences in T in which U_n doesn't occur for $n > k$.

- (Aug 2002) Show that $Th(\mathbb{R}, \mathbb{Q}, <)$ ($\mathbb{R}, \mathbb{Q}$ are the sets of reals and rationals) is ω-categorical and hence complete.

- (Jan 2002) Let $\mathcal{G} = (G, E)$ be a directed graph (E is a binary relation). The out-degree of a vertex in G is the number of outgoing edges from it. Show that the class of directed graphs all of whose vertices have finite out-degree is not axiomatizable.

- (Aug 2001) Suppose that a theory T has infinitely many distinct consistent completions. Show that T has a completion which is not finitely axiomatizable.

- (Aug 2001) List all complete theories of one equivalence relation on an infinite universe.

- (Jan 2001, Jan 2000) Show that addition is not definable (even with parameters) in $(\mathbb{N}, .)$ and $(\mathbb{Q}, <)$.

- (Aug 1996) Describe a finite theory T such that for each $n > 0$, T has a model of size 2^n and no model of any other finite size.

Logic SEP: Day 2

July 17, 2011

1 Examples on basic set theory

- (Jan 2011) If α and β are non zero ordinals, show that there is a largest ordinal δ that divides both of them. Here δ divides α if $\alpha = \delta\gamma$ for some γ.
- (Jan 2010) Let X be any set and $f : P(X) \to P(X)$ be order preserving, i.e., for any $A, B \in P(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$. Prove there exists $Y \subseteq X$ such that $f(Y) = Y$.
- (Aug 2009) For each prove or disprove:
(a) There exists a set D of reals with the same order type as the rationals which is a closed subset of the real number line.
(b) There exists a set D of reals with the same order type as the rationals which is discrete, i.e., no point of D is a limit point of D.
(c) There exists a set D of reals with the same order type as the rationals such that every point of D is a limit point of D but only from below and not above.
- (Jan 2009) In this problem, a real-valued function means a partial function F with $dom(F) \subseteq \mathbb{R}$ and $ran(F) \subseteq \mathbb{R}$; then, as a set, $F \subseteq \mathbb{R}^2$. Call such an F monotonic iff it satisfies either $\forall x_1, x_2 \in dom(F)(x_1 < x_2 \to F(x_1) \leq F(x_2))$ or $\forall x_1, x_2 \in dom(F)(x_1 < x_2 \to F(x_1) \geq F(x_2))$. Assuming the Continuum Hypothesis, prove that there is a real-valued function G such that $dom(G) = \mathbb{R}$ and $G \cap F$ is countable for all monotonic real-valued functions F.
- (Aug 2008) Prove of disprove: A linear order L is a well order iff every suborder of L is isomorphic to an initial segment of L.
- (Aug 2007) Show that every uncountable subset of reals contains an order isomorphic copy of rationals.
- (Jan 2007) If $(X, <)$ is totally ordered set, let $I(X, <)$ be the set of strictly increasing functions $f : X \to X$ (i.e. $f(x) < f(y)$ whenever $x < y$).
(a) Prove that $|I(\mathbb{R}, <)| = 2^\omega$ (where $<$ is the usual order).
(b) Give an example of a total order $(X, <)$ satisfying $|X| = 2^\omega$ and $|I(X, <)| = 2^{2^\omega}$.

- (Aug 2005) Let A be a set totally ordered by $<$, and assume that in A, there are no increasing or decreasing ω_1-sequences, and no subsets isomorphic to the rationals. Prove that A is countable.

- (Jan 2005) Prove that the Continuum Hypothesis is equivalent to the statement that there is a subset $A \subseteq \mathbb{R}$ of size ω_1 such that both A and $\mathbb{R} \backslash A$ meet every perfect subset of $\mathbb{R}$. A set is perfect iff it is closed and infinite and has no isolated points.

- (Aug 2004) Suppose $\{A_n | n \in \omega\}$ is a sequence of infinite sets. Prove there is a sequence $\{B_n | n \in \omega\}$ such that
(1) $B_n \cap B_m = \phi$ for each $n \neq m$, and
(2) $B_n \subseteq A_n$ and $|B_n| = |A_n|$ for each n.

- (Jan 2004) If $F \subseteq \lambda^\lambda$, say that F covers λ iff for all $\alpha, \beta < \lambda$ there is an $f \in F$ such that either $f(\alpha) = \beta$ or $f(\beta) = \alpha$. Let κ be any infinite cardinal Prove that κ^+ can be covered by a family of κ many functions, but not by any family of fewer than κ functions.

- (Aug 2003) Prove that there are countable $C_\alpha \subseteq \mathbb{R}$ for $\alpha < 2^\omega$ such that C_α and C_β are not isomorphic (with respect to the usual order on the real numbers) whenever $\alpha < \beta < 2^\omega$.

- (Aug 2002) Let κ be a cardinal with $\omega \leq \kappa \leq 2^\omega$. Prove that the following are equivalent:
(1) For all $X \subseteq \mathbb{R}$ with $|X| = \kappa$, there is a $q \in \mathbb{Q}$ such that $|X \cap (-\infty, q)| = |X \cap (q, +\infty)| = \kappa$.
(2) $cf(\kappa) > \omega$.

- (Jan 2002) Show that there is a subset of plane which meets every circle at exactly three points.

- (Jan 2001) Prove that every countable ordinal has the same order type as a closed set of reals.

- (Aug 1999) Let D_n, for $n \in \omega$, be a decreasing sequence of subsets of the plane $\mathbb{R}^2$. Assume that each D_n is dense (i.e., meets every nonempty open set). Prove that there is a dense subset E of plane such that each $E \backslash D_n$ is finite.

- (Aug 1998) Prove that there is an additive subgroup of the reals which is totally imperfect, ie. it and its complement intersect every uncountable closed set of reals.

- (Aug 1995) Let $\mathbb{R}$ be the set of real numbers. Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that f maps every perfect subset of $\mathbb{R}$ onto $\mathbb{R}$. We say $P \subseteq \mathbb{R}$ is perfect iff it is closed, non-empty, and has no isolated points (for example, the Cantor set, or any interval).

Logic SEP: Day 4

July 21, 2011

1 Examples on basic recursion theory

- (Aug 2010) Show that there is a computable group operation on ω whose center is not computable.
- (Aug 2009) For an abelian group G and prime p we say that G is p-divisible if for every $x \in G$ there is a $y \in G$ such that $py = x$. Prove that there is a computable $G \subseteq \mathbb{Q}$ which is a subgroup of $(\mathbb{Q}, +)$ such that $\{p | G$ is p-divisible $\}$ is not computable.
- (Jan 2009) Show that there is a computable equivalence relation on ω such that all of its equivalence classes are finite and the set of sizes of these equivalence classes is not computable.
- (Jan 2004) Let Σ be the theory of infinite abelian groups plus the axiom $\forall x(x + x + x = 0 \vee x + x = 0)$. Prove that Σ is decidable.
- (Aug 2003) Let $L = \{<, U\}$ where $<$ is a binary predicate and U is a unary predicate. Let T be an L-theory which says that $<$ is a dense linear order without end points and U is closed downward; i.e., for all $x < y$, $y \in U$ implies $x \in U$.
  (a) Up to isomorphism how many countable models does T have?
  (b) Show that T is decidable.
- (Aug 2003) Give an example of a computable $f : \omega \to \omega$ such that $\{f^n(0) | n \geq 1\}$ is not computable.
- (jan 2003) Let S be a uniformly computable sequence of computable sets; i.e., $S = \{A_m | m \in \omega\}$ where the map $(m, n) \mapsto \chi_{A_m}(n)$ is computable. Let B be the set of all boolean combination of sets in S. Then there is a computable set which is not in B.
- (Aug 2002) Show that the theory of $(\mathbb{C}, +, ., exp)$ is undecidable. Here, $exp(z) = e^z$.
- (Aug 2001) Let T be the theory of one infinite, coinfinite unary relation U. Show that T is decidable.

- (Aug 2000) Call a real number r computable iff the sequence of digits in the decimal representation of r is computable. Prove that there is a computable function $f : \omega \to \omega \setminus \{0\}$ such that $\sum \frac{1}{f(n)}$ is finite and not computable.

- (Aug 1998) Let T be a finitely axiomatizable theory in a finite language L. Assume that for each sentence θ of L, either $T \cup \{\theta\}$ has a finite model or $T \cup \{\theta\}$ is inconsistent. Prove that T is decidable.

- (Aug 1996) Let $L = \{<\} \cup \{c_q | q \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rationals. Let A be the natural model for L; that is, $A = \mathbb{Q}$, each c_q is interpreted as q, and $<$ is interpreted as the usual order on rationals. Prove that the theory of A is decidable.

- (Aug 1995) Let L be the language consisting of $=$, two binary functions, $+, .$, and one unary function, f. Let A be the structure whose domain of discourse is the set of real numbers, where $+$, . are interpreted as the usual addition and multiplication, and f is interpreted as $f(x) = \sin x$. Prove that the theory of A is undecidable.

- (Jan 1994) For any theory T let F be the set of all sentences in the language of T which are true in some finite model of T. Assume the language of T is recursive.
(a) If T is finitely axiomatizable show F is recursively enumerable.
(b) If T is decidable show F is recursively enumerable. Warning: The language of T might be infinite and even if it is finite, T might not be finitely axiomatizable.
(c) Give an example of a recursively axiomatizable T in the language of pure equality such that F is not recursively enumerable.

- (Jan 1987) Let $L = 0$: the empty language. So L-formulas can only use equality symbol. Show that the set of all L-sentences which are true in every L-structure is computable.

Logic SEP: Day 5

July 25, 2011

1 Quiz on elementary problems

- (M1) Is the theory of $(\mathbb{R}, <, \mathbb{Z})$ ω-categorical? Justify?
- (M2) Give an example of a complete theory in a countable language which has exactly 3 countable models.
- (C1) True or False? Every finite extension of a decidable theory is decidable.
- (C2) True or false? Every c.e. theory is computably axiomatizable.
- (S1) Let κ be an infinite cardinal. Let λ be the least cardinal for which $\kappa^\lambda > \kappa$. Show that λ is regular.
- (S2) Working in ZF, show that the following statements are equivalent:
  (1) Every set can be totally ordered.
  (2) For every set X, $(P(X), \subseteq)$ has maximal chain; i.e., a maximal linearly ordered subset.

Logic SEP: Day 2

July 15, 2012

1 Quiz 1: Do one problem each from sections A and B

- Section A

  A linear order L is called homogeneous if every finite partial order isomorphism on L can be extended to an automorphism of L. Show that there are homogeneous linear orders of every infinite cardinality.

  Show that the class of well orderings is not axiomatizable.

  Give an example of a theory in a finite language that is ω_1-categorical but not ω-categorical.

  Can there be a theory T such that T has precisely ω many pairwise non isomorphic models?

- Section B

  True or False? Justify.
  (a) The closure of an isolated set of reals is countable.
  (b) If X is a set of reals each of whose subsets has an isolated point, then X is countable.

  Show that there is a countable family of functions $\{f_n : \mathbb{R} \to \{0,1\}\}$ such that for every function $f : \mathbb{R} \to \{0,1\}$ and every finite set $F \subseteq \mathbb{R}$ there exists an n such that $f \restriction F = f_n \restriction F$.

  Show that there is a function $f : [\mathbb{R}]^2 \to \{0,1\}$ such that for every uncountable $X \subseteq \mathbb{R}$, $f \restriction [X]^2$ is not constant. Here $[X]^2$ denotes the collection of all subsets of X of size 2.