Part 1 of Martin's Conjecture for Order Preserving Functions

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TL;DR

Martin's conjecture: classifies all the "natural" functions on the Turing degrees.

Our result: if a "natural" function on the Turing degrees satisfies an additional condition (being order-preserving) then it is either eventually constant or eventually increasing.

Where are the natural intermediate degrees?

It is easy to construct Turing degrees in-between 0 and $0'$. So why are all the undecidable problems that come up in mathematics at least as hard as the halting problem?

Martin's conjecture provides a partial explanation of this phenomenon.

The idea is that natural problems can be used to define operators on the Turing degrees.

Propaganda for Martin's conjecture

A "natural" undecidable problem A should be

- **Relativizable:** for each oracle $X \subseteq \mathbb{N}$, we have a version of the problem A relative to X, i.e. A defines an operator $X \mapsto A(X)$
- **Degree invariant:** equivalent oracles give equivalent versions of the problem, i.e. if $X \equiv_T Y$ then $A(X) \equiv_T A(Y)$

The point: A induces a function on the Turing degrees.

Martin's conjecture: super informal version

Very loosely, Martin's conjecture says that the only "natural" functions on the Turing degrees are the iterates of the Turing jump.

Of course, it looks like we've now attempted to explain a vague statement about natural Turing degrees by making a vague conjecture about natural functions on the Turing degrees.

The value of Martin's conjecture is the way it makes this precise, which I'll explain next.

What are the natural functions on the Turing degrees?

$$\text{Constantly zero:} \quad x \mapsto 0$$

$$\text{Identity:} \quad x \mapsto x$$

$$\text{Jump:} \quad x \mapsto x'$$

$$\text{Double jump:} \quad x \mapsto x''$$

$$\dots$$

$$\text{Hyperjump:} \quad x \mapsto \mathcal{O}^x$$

$$\dots$$

What are the natural functions on the Turing degrees?

Intuitively: just the transfinite iterates of the jump.
But it is easy to construct others.

Example 1: For every x there is y such that

$$x <_T y <_T x'.$$

Use choice to pick one such y for each x.

What are the natural functions on the Turing degrees?

Intuitively: just the transfinite iterates of the jump.
But it is easy to construct others.

Example 2: Fix a Turing degree z and define

$$f(x) = \begin{cases} 0 & \text{if } x \not\geq_T z \\ x' & \text{if } x \geq_T z. \end{cases}$$

What are the natural functions on the Turing degrees?

Idea of Martin's conjecture: Exclude these types of examples

- Remove the axiom of choice
- Only look at the behavior of functions "in the limit"

What does "in the limit" mean?

Definition: For f, g functions on the Turing degrees

- $f \equiv g$ if there is some z such that

$$x \geq_T z \implies f(x) = g(x)$$

"$f = g$ on the cone above z"

- $f \leq g$ if there is some z such that

$$x \geq_T z \implies f(x) \leq_T g(x)$$

"$f \leq g$ on the cone above z"

What does "in the limit" mean?

More generally: For A a set of Turing degrees

- A has measure 1 if there is some z such that

$$x \geq_T z \implies x \in A$$

"A contains a cone"

- A has measure 0 if there is some z such that

$$x \geq_T z \implies x \notin A$$

"A is disjoint from a cone"

Fact: This forms a $\{0, 1\}$-valued measure on the Turing degrees, called Martin measure

Removing the axiom of choice

Statement of Martin's conjecture removes choice but adds the axiom of determinacy (AD)

Why?

- **Philosophical reason:** If you can't construct a function in ZF + AD then you also can't construct it in ZF
- **Practical reason:** AD allows you to prove structural theorems, gives some hope of classifying all functions on the Turing degrees
- **Philosophical reason 2:** Assuming large cardinal hypotheses, AD typically holds for "definable sets" (e.g. sets in $L(\mathbb{R})$)

Removing the axiom of choice

Statement of Martin's conjecture removes choice but adds the axiom of determinacy (AD)

Assuming the axiom of determinacy:

Fact: The Martin measure is an ultrafilter.

Fact, restated: Every set of Turing degrees either contains a cone or is disjoint from a cone

Fact, restated again: If for every x there is $y \geq_T x$ such that $y \in A$ then A contains a cone ("if A is cofinal then A contains a cone")

Philosophy of using determinacy in computability

Principle 1: Describe what you want, show it is cofinal, and let determinacy do the rest.

Example (jump inversion via nuclear flyswatter):
There is some z such that for each $x \geq_T z$ there is y with $y' \equiv_T x$.

Proof: Let $A = \{x \mid \exists y \, (y' = x)\}$. This set is cofinal since for each x, $x' \geq_T x$ and has this property. So A contains a cone.

This example is kind of absurd because we already know that this property holds on the cone above $0'$

Philosophy of using determinacy in computability

Principle 2: If the union of countably many sets is cofinal then so is at least one of the sets.

Example: If f is below a constant function then it is constant on a cone (i.e. equivalent to a constant function).

Proof: Assume that for all x, $f(x) \leq_T c$. There are countably many degrees below c and the union of the their preimages is the entire Turing degrees. So for some y, $f^{-1}(\{y\})$ is cofinal and so by determinacy it contains a cone.

Martin's Conjecture

Statement of Martin's conjecture: Assuming the axiom of determinacy

(1) Every function on the Turing degrees is either equivalent to a constant function or greater than or equal to the identity function

(2) The (equivalence classes of) functions which are increasing form a well-order where the successor is given by the jump (i.e. successor of f is $x \mapsto f(x)'$)

Disclaimer: Martin's conjecture is usually stated in terms of Turing-invariant functions on 2^ω. Assuming $AD_\mathbb{R}$ or AD^+ (two strengthenings of the axiom of determinacy), this is equivalent.

Some Past Results

Theorem (Slaman and Steel 1980's): Part 1 of Martin's conjecture holds for functions below the identity.

Restated: If $f(x) \leq_T x$ for all x then f is either constant on a cone or equal to the identity on a cone

Some Past Results

Definition: If f is a function on the Turing degrees, f is order-preserving if for all x and y

$$x \leq_T y \implies f(x) \leq_T f(y)$$

Theorem (Slaman and Steel 1980's): Part 2 of Martin's conjecture holds for order-preserving functions which are below the hyperjump.

Restated: Equivalence classes of order-preserving functions which are above the identity and below the hyperjump form a well-order with successor given by the jump.

Our Main Results

Theorem (L. and Siskind): Part 1 of Martin's conjecture holds for order-preserving functions.

Restated: An order-preserving function on the Turing degrees is either constant on a cone or increasing on a cone.

Rules out "sideways" order-preserving functions (i.e. functions f for which $f(x)$ is incomparable to x)

Our Main Results

A function f on the Turing degrees is measure-preserving if for all x there is some y such that

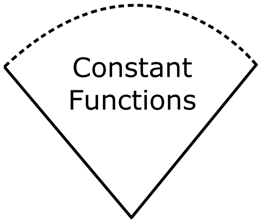
$$z \geq_T y \implies f(z) \geq_T x$$

i.e. f is greater than every constant function.
or "f goes to infinity in the limit"

Theorem (L. and Siskind): Part 1 of Martin's conjecture holds for measure-preserving functions.

Restated: A function which is above every constant function is also above the identity.

Picture of functions on the Turing degrees



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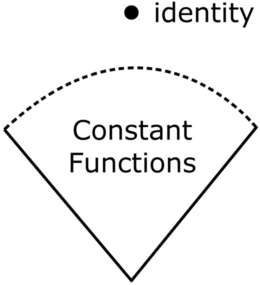

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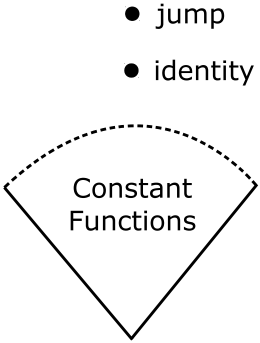

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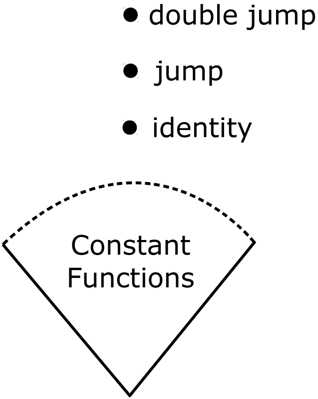

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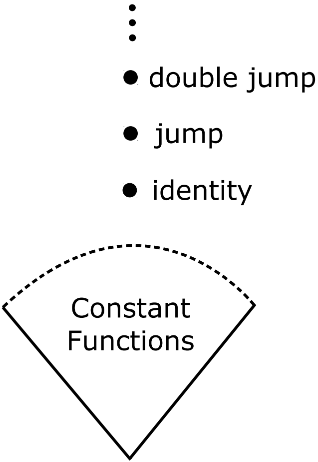

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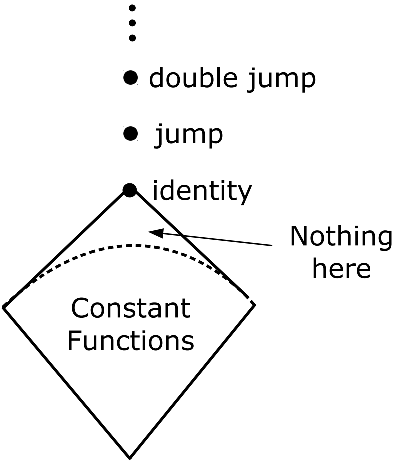

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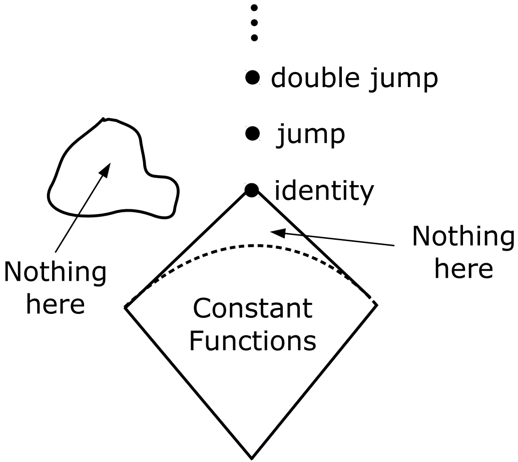

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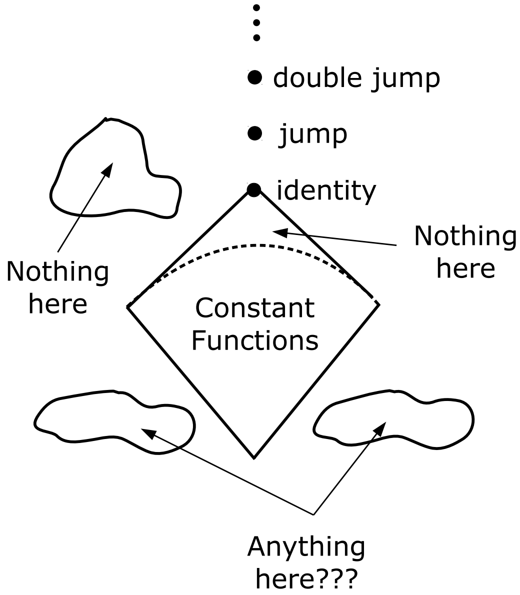

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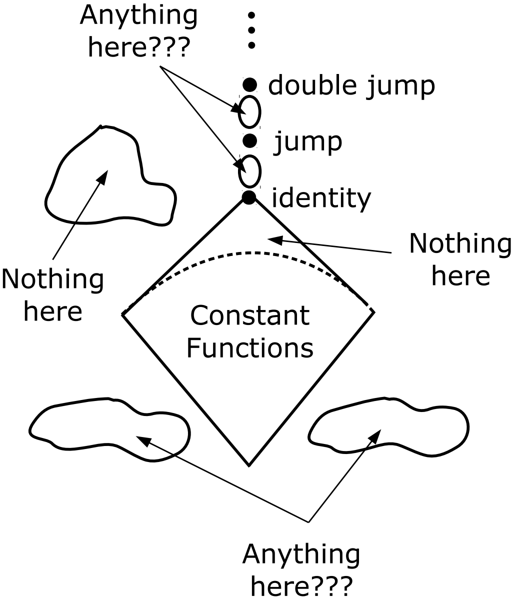


High Level Overview

(1) Identify a combinatorial condition (being measure-preserving) that is enough to prove part 1 of Martin's conjecture

(2) Show that every order-preserving function is either constant on a cone or measure-preserving using a new basis theorem for perfect sets (I will skip this)

(3) Show that every measure-preserving function is above the identity (I will focus on this)

High Level Overview

How does the proof for measure-preserving functions work?

(1) General framework is a basic topological fact about continuous, injective functions

(2) To apply this in our case, study the structure of measure-preserving functions under determinacy

(3) In more detail: use determinacy to get certain auxiliary functions associated to a measure preserving function (basically a Skolem function witnessing that it is measure-preserving, and the inverse of this Skolem function)

A basic fact of topology

A basic theorem in topology: If $F : X \to X$ is a continuous, injective function on a compact, Hausdorff topological space X then F has continuous inverse range$(F) \to X$.

Computability theory version: If $F : 2^\omega \to 2^\omega$ is a computable injective function then for all x, $F(x)$ can compute x.

Key idea: To show a function f is above the identity, it is enough to find a computable, injective function which is below f.

Some technicalities

Key idea: To show a function f is above the identity, it is enough to find a computable, injective function which is below f.

Actually, we need to use a more refined version of this basic strategy.

Computability theory version, refined: If T is a pointed perfect tree and $F : [T] \to 2^\omega$ is a computable injective function then for all x in $[T]$, $F(x) \oplus T$ can compute x.

What is a pointed perfect tree?

Definition: A pointed perfect tree is a perfect binary tree T such that all infinite paths through T compute T

Notation: $[T]$ = set of infinite paths through T

If T is a pointed perfect tree then every Turing degree above T has a representative in $[T]$

Theorem (AD): If $A \subseteq 2^\omega$ is such that for all x there is $y \in A$ with $y \geq_T x$ (i.e. A is cofinal) then there is a pointed perfect tree T such that $[T] \subseteq A$

The point: We can use determinacy for subsets of 2^ω in addition to sets of Turing degrees

Proof strategy in more detail

- Start with a measure-preserving function f
- Find a pointed perfect tree T and a computable injective function g on $[T]$ which is below f.
- So for every $x \in [T]$, $g(x) \oplus T$ can compute x
- Since f is measure-preserving, $f(x)$ can eventually compute T
- So $f(x)$ can eventually compute $g(x) \oplus T$ and hence also x

To see how to find T and g, we need to understand better what we can do with measure-preserving functions under determinacy.

What are measure-preserving functions again?

Definition: A function f on the Turing degrees is measure-preserving if for all x there is some y such that

$$z \geq_T y \implies f(z) \geq_T x$$

"f goes to infinity in the limit"

This definition naturally suggests looking at the Skolem function which witnesses that f is measure-preserving.

More about measure-preserving functions

Definition: If f is a measure-preserving function, call $g : 2^\omega \to 2^\omega$ a modulus for f if for all x,

$$z \geq_T g(x) \implies f(z) \geq_T x.$$

Call g an increasing modulus for f if in addition we have $g(x) \geq_T x$.

It may seem obvious that every measure-preserving functions has a modulus. But you are probably using the axiom of choice.

However, it is also true under determinacy! (By a uniformization theorem)

Disclaimer: needs $\mathsf{AD}_\mathbb{R}$ or AD^+

How to use the modulus

Remember: We are trying to find a computable injective function which f computes. Here's how we find it.

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g (like in the jump inversion via nuclear flyswatter example)

How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g

Explanation: suppose $h : \mathrm{range}(g) \to 2^\omega$ is an inverse for g—i.e. $g(h(x)) = x$.

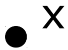
- h is injective: if $h(x) = h(y)$ then

$$x = g(h(x)) = g(h(y)) = y$$

- h is computable: $h(x) \leq_T g(h(x)) = x$ because g is increasing
- f is above h: $x \geq_T g(h(x))$ so $f(x) \geq_T h(x)$

How to use the modulus

Fix an increasing modulus g for f. We get the function we want by using determinacy to invert g



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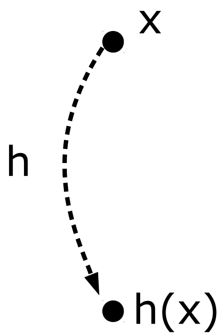

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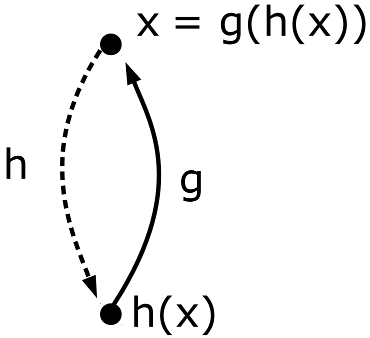

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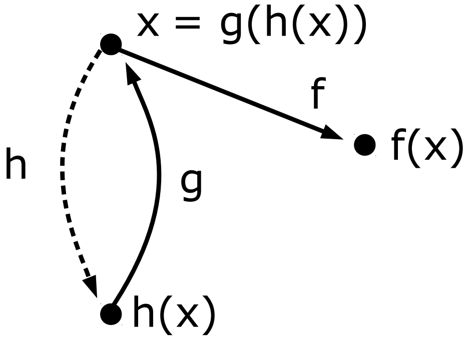

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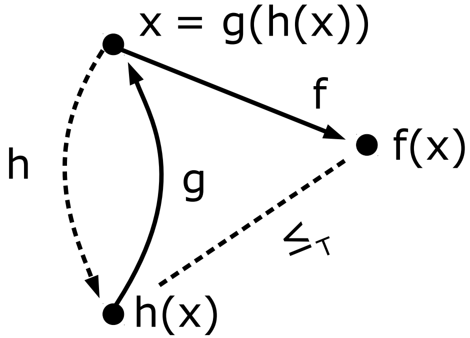

How is this possible??

This function h seems like exactly the kind of thing that's supposed to be ruled out by Martin's conjecture! It's decreasing and injective (so it's not constant on any cone). Why is this possible?

Answer: h is a function on 2^ω and there is no guarantee it is Turing invariant (i.e. even if $x \equiv_T y$ we may have $h(x) \not\equiv_T h(y)$) so it does not induce a function on Turing degrees.

This is a key point in our proof: you can study a Turing invariant function by relating it to a non-Turing invariant function you get with determinacy

How to use determinacy to invert the modulus

Suppose g is an increasing modulus for f. Since g is increasing, the set $A = \{x \mid \exists y\,(g(y) = x)\}$ is cofinal (just like jump inversion example). Define

$$A_e = \{x \mid \Phi_e(x) \text{ is total} \wedge g(\Phi_e(x)) = x\}$$

Since g is increasing, any y such that $g(y) = x$ must be computable from x. So we have

$$A = \bigcup_{e \in \mathbb{N}} A_e.$$

Hence one of the A_e's is cofinal and so contains a pointed perfect tree. Set $h = \Phi_e$ for this e

An alternative proof

The proof we just saw uses more determinacy than just AD. Let's see why.

The basic strategy was to find a computable injective function g which is below f. The g we used was injective because it was the inverse of another function. But to find that other function, we had to use more determinacy than just AD.

Question: Is there another way to find an injective function g?

Answer: Yes! (but only for order-preserving functions)

An alternative proof

The basic strategy is the same as before: find a computable injective function g which is below f.

Previously, the g we found was injective because it was the inverse of another function. What can we do instead?

Fact (Spector-style perfect tree dichotomy): If T is a pointed perfect tree and g is a computable function on $[T]$ then either g is injective on a pointed perfect subtree of T or g is constant on a pointed perfect subtree of T.

Upshot: Just need to find g which is not eventually constant.

An alternative proof

The basic strategy is the same as before: find a computable injective function g which is below f.

It's enough to find g which is not eventually constant.

We will ensure that by picking g which preserves an ordinal invariant

Ordinal invariants

Definition: An ordinal invariant is a function from the Turing degrees to the ordinals

Prototypical example: $x \mapsto \omega_1^x$, where ω_1^x is the least ordinal with no presentation computable by x

Ordinal invariants

Fact: If α is an ordinal invariant, then α is order preserving on a cone—i.e. there is some z such that for all $x, y \geq_T z$

$$x \leq_T y \implies \alpha(x) \leq_T \alpha(y)$$

Theorem: If f is a measure preserving function and α is an ordinal invariant then f preserves α on a cone—i.e. there is some z such that

$$x \geq_T z \implies \alpha(x) \leq_T \alpha(f(x)).$$

Ordinal invariants

Theorem (abbreviated): $\alpha(x) \leq f(\alpha(x))$ on a cone

Proof: Suppose not. Then by determinacy, $\alpha(f(x)) < \alpha(x)$ on the cone above some z.

Key point: If f is measure-preserving then so are

$$f \circ f, f \circ f \circ f, \ldots$$

So we can find x large enough that x, $f(x)$, $f(f(x))$, ... are all above z. We have

$$\alpha(x) > \alpha(f(x)) > \alpha(f(f(x))) > \ldots$$

which is a descending sequence in the ordinals!

Brief sketch of alternative proof

- Define some ordinal invariant α
- Use determinacy to get a pointed perfect tree T and a computable function g on $[T]$ which is below f and such that $g(\alpha(x)) \geq \alpha(x)$.
- g can't be constant on any pointed perfect tree because you can always make $\alpha(x)$ increase by increasing x

Fact that measure-preserving functions preserve ordinal invariants is used in finding g.

A suggestive fact

Definition (ergodic theory): If $(X, \mathcal{A}, \mu)$ is a measure space and $F : X \to Y$ is a function then the pushforward of μ by F, written $F_*(\mu)$ is the measure on Y given by $F_*(\mu)(A) = \mu(F^{-1}(A))$.

Definition (ergodic theory): If $(X, \mathcal{A}, \mu)$ is a measure space and $F : X \to X$ is a measurable function then F is called measure-preserving if $F_*(\mu) = \mu$.

Fact: A function f on the Turing degrees is measure-preserving in our sense if and only if

$$f_*(\text{Martin measure}) = \text{Martin measure}$$

A suggestive fact

Fact: A function f on the Turing degrees is measure-preserving in our sense if and only if

$$f_*(\text{Martin measure}) = \text{Martin measure}$$

An unexpected conseuqence: Our proof for measure-preserving functions implies that part 1 of Martin's conjecture is equivalent to the statement "the Martin measure is minimal in the Rudin-Keisler order on ultrafilters on the Turing degrees"

The view from the ultrapower

Since the Martin measure is an ultrafilter, we can use it to take ultrapowers of structures.

- Since the Martin measure is countably complete, the ultrapower of the ordinals is well-founded.
- An ordinal invariant is a (representative of an) element of this ultrapower
- A measure-preserving function induces an embedding on the ultrapower
- Embeddings on well-ordered sets are always non-decreasing

This gives an alternative proof of the theorem about ordinal invariants!

A new direction?

Ultrapowers by the Martin measure have been studied in the context of the descriptive set theory of $L(\mathbb{R})$

In that context, there is also the concept of a "generic ultrapower" using a notion of forcing whose conditions are pointed perfect trees.

Does all of this mean that we can understand the structure of functions on the Turing degrees by applying tools from descriptive set theory? Do things become clearer if we work in the ultrapower by the Martin measure?

Afterword: New basis theorem for perfect sets

Theorem (L.): If A is a perfect subset of 2^ω, B is a countable dense subset of A and x computes every element of B then for every y there are $z_0, z_1, z_2, z_3 \in A$ such that

$$x \oplus z_0 \oplus z_1 \oplus z_2 \oplus z_3 \geq_T y$$

Proof is pretty much pure computability theory.