# Noncomputable Coding, Density, Stochasticity

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September 15th, 2020

XXV(iii) Midwest Computability Seminar
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# Asymptotic Density

Recall that the (asymptotic) density of $X$ is

$$\rho(X) = \lim_{n\to\infty} \frac{|X \upharpoonright n|}{n}$$

if this limit exists. $\overline{\rho}(X)$ and $\underline{\rho}(X)$, the upper and lower density, are the limit superior and limit inferior respectively.

# Intrinsic Density

Intrinsic Density is the computable permutation invariant form of asymptotic density, that is $A$ has intrinsic density $\alpha$ if $\rho(\pi(A)) = \alpha$ for all computable permutations $\pi$. We denote it by $P(A)$ if it exists. $A$ is intrinsically small if $P(A) = 0$.

The absolute upper density, $\overline{P}(A)$, is the limit superior of $\rho(\pi(A))$ over all computable permutations $\pi$. The absolute lower density, $\underline{P}(A)$, is the limit inferior.

# Randomness

What values of $r \in [0, 1]$ are achieved as the intrinsic density of a set?

The short answer is everything: 0 and 1 are well known. If $\mu_r$ is the Bernoulli measure with parameter $r$ (i.e. the $r$-biased coin flip measure), then standard arguments show that every $\mu_r$-1-random set has intrinsic density $r$. (It also holds for Schnorr randomness, but the proof takes more work.)

# Motivation

Can we use sets of a given intrinsic density to understand those of a different one?

As we shall see, sets with intrinsic density $r$ are a strict super set of the $\mu_r$-randoms and have fundamentally different closure properties as a class. (The join preserves intrinsic density.) We shall introduce noncomputable coding methods, the `into` and `within` operations, to change intrinsic densities and construct an example for every real $r \in (0,1)$ without appealing to $\mu_r$-randomness.

# Changing Density via Noncomputable Coding

To understand intrinsic density, we set out to find some method of combining sets $A$ and $B$ such that the intrinsic density of the resulting set is some function of the intrinsic densities of $A$ and $B$. We shall see that the `into` operation achieves this, but first we will demonstrate why classical methods do not work.

# Failure of the Join

**Theorem**

$P(A \oplus B) = \alpha$ *if and only if* $P(A) = P(B) = \alpha$.

**Proof Sketch.**

- ($\Rightarrow$) Suppose $P(A) \neq P(B)$ as witnessed by $\pi_A$ and $\pi_B$. Then we can construct a permutation which makes the evens look like almost everything after applying $\pi_A$ to show that the density of $A \oplus B$ under this permutation is the same as the density of $\pi_A$. Then we do the same for $B$.
- ($\Leftarrow$) Harder.

□

# The Into Operation

## Definition

Given two sets

$$A = \{a_0 < a_1 < a_2 < \dots\}$$

and

$$B = \{b_0 < b_1 < b_2 < \dots\}$$

we define the set $B \triangleright A$, or "$B$ into $A$," to be

$$\{a_{b_0} < a_{b_1} < a_{b_2} < \dots\}$$

This will be a fundamental tool for changing densities.

# Examples

- If $E$ is the set of evens, then $E \triangleright E$ is the set of multiples of 4: $e_n = 2n$, so

$$e_{e_n} = e_{2n} = 2(2n) = 4n$$

- If $O$ is the set of odds, then $E \triangleright O$ is the set of naturals congruent to 1 mod 4: $o_n = 2n + 1$ and $e_n = 2n$, so

$$o_{e_n} = o_{2n} = 2(2n) + 1 = 4n + 1$$

- For any $A$ and $B$,

$$A \oplus B = (A \triangleright E) \sqcup (B \triangleright O)$$

# Basic Properties

- $A = A \triangleright \omega$
- $A = \omega \triangleright A$
- $B \triangleright A \subseteq A$
- $(B \triangleright A) \sqcup (\overline{B} \triangleright A) = A$
- $\triangleright$ is associative

# Example Application

While our main goal is to use `into` to understand density, it is useful as a tool for noncomputable coding in its own right.

## Lemma

*If $\mathcal{P} \subseteq 2^\omega$ is closed under subsets and self join, i.e. $X \oplus X \in \mathcal{P}$ for all $X \in \mathcal{P}$, then the $\mathcal{P}$-degrees are closed upwards.*

## Proof.

If $A \geq_T B$ and $B \in \mathcal{P}$, then $B \oplus (A \triangleright B) \in \mathcal{P}$. Furthermore, $A \equiv_T B \oplus (A \triangleright B)$ as

$$A = \{n : 2n \text{ and } 2n+1 \in B \oplus (A \triangleright B)$$

□

As a corollary, we obtain a simple proof of the classical result due to Martin and Miller that the hyperimmune degrees are closed upwards.

# A New Application

Astor showed that the degrees of intrinsically small sets are exactly the high or DNC degrees. This proof used the result of Jockusch that the degrees of any collection which is closed under subsets and has an arithmetical element are closed upwards.

Most of this proof relativizes. The only problem is the result of Jockusch: there are sets for which there is no arithmetical intrinsically small set. (For example, $\emptyset^{(\omega)}$.) However, in this instance we can replace this with our previous result to prove the full relativized version of this theorem.

Note that in general this is not an improvement of Jockusch's result: The cohesive sets, Jockusch's original motivation, are not closed under self join.

# Into and Asymptotic Density

## Lemma

- $\overline{\rho}(B \triangleright A) \leq \overline{\rho}(A)\overline{\rho}(B)$
- $\underline{\rho}(B \triangleright A) \geq \underline{\rho}(A)\underline{\rho}(B)$

## Proof Sketch.

$$\overline{\rho}(B \triangleright A) = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} = \limsup_{n\to\infty} \frac{n+1}{a_{b_n}+1} \cdot \frac{b_n+1}{b_n+1} \leq$$

$$(\limsup_{n\to\infty} \frac{b_n+1}{a_{b_n}+1})(\limsup_{n\to\infty} \frac{n+1}{b_n+1}) \leq \overline{\rho}(A)\overline{\rho}(B)$$

□

## Corollary

$\rho(B \triangleright A) = \rho(B)\rho(A)$

# $\mathtt{Into}$ and Intrinsic Density

For a set $X$, $P_X$ represents intrinsic density relative to $X$, i.e. invariance under all $X$-computable permutations.

**Theorem**

*If* $P(A) = \alpha$ *and* $P_A(B) = \beta$*, then* $P(B \triangleright A) = \alpha\beta$*.*

# Is this Best Possible?

**Theorem**

*If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(B \triangleright A) = \alpha\beta$.*

We see that the conditions cannot be weakened in general.

- $P(A) = \alpha$ is a necessary requirement, as $P_A(\omega) = 1$ for all $A$ and $\omega \triangleright A = A$.
- $P_A(B) = \beta$ is necessary: Let $P(A) = \frac{1}{2}$. Then $P(A \oplus \overline{A}) = \frac{1}{2}$, but $A \triangleright (A \oplus \overline{A}) = A \oplus \emptyset$ does not have intrinsic density

To prove the theorem, we first need to introduce another operation.

# The `Within` Operation

The `into` operation has a natural dual.

## Definition

Given two sets

$$A = \{a_0 < a_1 < a_2 < \dots\}$$

and

$$B = \{b_0 < b_1 < b_2 < \dots\}$$

we define the set $B \triangleleft A$, or "$B$ within $A$", to be $\{n : a_n \in B\}$. In other words, $B \cap A$ is some subset of $A$, so there is some $X$ such that $X \triangleright A = B \cap A$. In this case, $B \triangleleft A = X$.

# Example

- Let $T$ be the multiple of three. Then $T \triangleleft E = T$:

$$T \triangleleft E = \{n : 2n \in T\} = \{n : 6|2n\} = \{n : 3|n\}$$

- $T \triangleleft O$ is the set of naturals congruent to 1 mod 3:

$$T \triangleleft O = \{n : 2n + 1 \in T\} = \{n : 2n \equiv 2 \bmod 3\} = \{n : n \equiv 1 \bmod 3\}$$

# Within: Basic Properties

- $\omega = A \triangleleft A$
- $(B \triangleleft A) \sqcup (\overline{B} \triangleleft A) = \omega$
- If $B \subseteq A$, then $(B \triangleleft A) \triangleright A = B \cap A = B$.
- $\triangleleft$ is not associative: Let $E$ be the evens, $O$ the odds, and $N$ the set of naturals congruent to 2 mod 4. Then

$$(O \triangleleft N) \triangleleft E = \emptyset \triangleleft E = \emptyset$$

but

$$O \triangleleft (N \triangleleft E) = O \triangleleft O = \omega$$

# Within and Intrinsic Density

**Lemma**

*If C is computable and $P(A) = \alpha$, then $P(A \triangleleft C) = \alpha$.*

The proof of this result informs us how to prove our critical theorem.

# Proof Sketch

**Theorem**

*If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(B \triangleright A) = \alpha\beta$.*

Fix a computable permutation $\pi$. Then $\rho(\pi(A)) = \alpha$. Construct a computable permutation $\pi_A$ such that $\rho(\pi_A(B)) = \rho(\pi(B \triangleright A) \triangleleft \pi(A))$. As $\rho(\pi_A(B)) = \beta$, we can use the lemma for asymptotic density to see that

$$\rho(\pi(B \triangleright A)) = \rho((\pi(B \triangleright A) \triangleleft \pi(A)) \triangleright \pi(A)) = \rho(\pi(B \triangleright A) \triangleleft \pi(A))\rho(\pi(A)) = \alpha\beta$$

# Intersection

## Corollary

*If $P(A) = \alpha$ and $P_A(B) = \beta$, then $P(A \cap B) = \alpha\beta$.*

## Proof.

As $P_A(B) = \beta$, $P_A(B \triangleleft A) = \beta$ by the relativized form of the above lemma. Therefore applying the previous theorem yields

$$P((B \triangleleft A) \triangleright A) = P(A \cap B) = \alpha\beta$$

$\square$

# Union

## Lemma (Jockusch and Schupp)

*If there is a countable sequence* $\{S_i\}_{i\in\omega}$ *of disjoint sets such that* $P(S_i)$ *exists for all* $i$ *and*

$$\lim_{n\to\infty} \overline{P}\Big(\bigsqcup_{i>n} S_i\Big) = 0$$

*then*

$$P\Big(\bigsqcup_{i\in\omega} S_i\Big) = \sum_{i=0}^{\infty} P(S_i)$$

# Powers of Two

## Lemma

*There is a countable, disjoint sequence of sets* $\{A_i\}_{i\in\omega}$ *such that* $P(A_i) = \frac{1}{2^{i+1}}$. *Furthermore,* $\lim_{n\to\infty} \overline{P}(\bigsqcup_{i>n} A_i) = 0$.

## Proof Sketch.

Let $X$ be 1-Random. Then by the general form of Van Lambalgen's theorem and the fact that 1-Randoms have intrinsic density $\frac{1}{2}$, the columns $X^{[n]}$ of $X$ give us countably many sets all with intrinsic density relative to the rest. Then define $B_0 = \omega$, $A_n = \overline{X^{[n]}} \triangleright B_n$, and $B_{n+1} = X^{[n]} \triangleright B_n$. □

# Arbitrary Intrinsic Density

Let $r \in (0,1)$ and $\{A_i\}_{i\in\omega}$ be as in the previous lemma. We identify $B_r$ with the set of bits which are 1 in the binary expansion of $r$. Then $\bigcup_{i\in B_r} A_i$ will have intrinsic density $\Sigma_{i\in B_r} \frac{1}{2^{i+1}} = r$.

## Corollary

*If $r \in (0,1)$ computes a 1-Random, then $r$ computes a set of intrinsic density $r$.*

# Stochasticity

With stochasticity, countably many 0-1 valued coins have been flipped and are lying in a row. (We think of this sequence as a set, i.e. the set of all $n$ such that the $n$-th coin is a 1.) Our goal is to try and successfully select an infinite subsequence which has a different ratio of 1's to 0's than the original sequence. A set is "stochastic" if we cannot do this. This depends on how we are allowed to select our subsequence.

# Injection Stochasticity

Formally, $X$ is $r$-injection stochastic if $\rho(i^{-1}(X)) = \rho(\{n : i(n) \in X\}) = r$ for all total computable injections $i$.

The idea here is that given a sequence of coin flips $X$, we cannot build an infinite sequence $Y$ of different density by picking and choosing coins from $X$ in some computable (i.e. without knowing the values of $X$) fashion (represented by $i$.) Every coin can only be picked once ($i$ must be an injection), but some coins may be discarded ($i$ need not be a surjection.) We can index coins out of order, i.e. pick the second coin before the first, but every index must eventually get a coin ($i$ must be total.)

# Illustration for Injection Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\dots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| X | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | $\dots$ |
| $i^{-1}(n)$ | 2 | 1 | $\uparrow$ | 4 | 0 | $\uparrow$ | $\uparrow$ | $\uparrow$ | 3 | $\dots$ |
| Y | 1 | 1 | 0 | 0 | 1 | $\dots$ | | | | |

# Connecting Stochasticity and Density

**Lemma (Astor)**

*A set $A$ is $r$-injection stochastic if and only if it has intrinsic density $r$.*

# Church Stochasticity

Our sequence of coin flips has been covered by cups. With no information, we decide if we want to add the first coin to our subsequence or not. We then remove the first cup and check the value of the coin before adding it to our subsequence or discarding it based on our original choice. We continue adding to our subsequence, deciding whether or not to include the $n+1$-st coin in our subsequence using only the information about the result of the first $n$ flips.

# Church Stochasticity (Cont.)

Formally, a selection function is a function $f : 2^{<\omega} \to \{0,1\}$. This induces a map $\hat{f}$ on Cantor space defined via $\hat{f}(X) = \{n : f(X \upharpoonright n) = 1\}$. We abuse notation and let $f(X) = \hat{f}(X)$.

Then $X$ is $r$-Church stochastic if for all total computable selection functions $f$, either $f(X)$ is finite or $\rho(X \triangleleft f(X)) = r$. Recall that

$$X \triangleleft f(X) = \{n : p_{f(X)}(n) \in X\}$$

In other words, $X \triangleleft f(X)$ is the set of $n$ such that the $n$-th coin selected from $X$ by $f$ is a 1.

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(\emptyset)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(1)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(10)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(101)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(\emptyset)$ |
| | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(1)$ |
| | 1 | 0 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(10)$ |
| | 1 | 0 | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(101)$ |
| | 1 | 0 | 1 | 0 | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(1010)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ... | $f(\emptyset)$ |
| | 1 | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ... | $f(1)$ |
| | 1 | 0 | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ... | $f(10)$ |
| | 1 | 0 | 1 | ◦ | ◦ | ◦ | ◦ | ◦ | ◦ | ... | $f(101)$ |
| | 1 | 0 | 1 | 0 | ◦ | ◦ | ◦ | ◦ | ◦ | ... | $f(1010)$ |
| | 1 | 0 | 1 | 0 | 0 | ◦ | ◦ | ◦ | ◦ | ... | $f(10100)$ |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(\emptyset)$ |
| | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(1)$ |
| | 1 | 0 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(10)$ |
| | 1 | 0 | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(101)$ |
| | 1 | 0 | 1 | 0 | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(1010)$ |
| | 1 | 0 | 1 | 0 | 0 | ∘ | ∘ | ∘ | ∘ | … | $f(10100)$ |
| | | | | | … | | | | | | |
| $f(X)$ | 0 | 1 | 1 | 0 | 1 | 0 | … | | | | |

# Illustration for Church Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\ldots$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(101)$ |
| | 1 | 0 | 1 | 0 | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(1010)$ |
| | 1 | 0 | 1 | 0 | 0 | ○ | ○ | ○ | ○ | $\ldots$ | $f(10100)$ |
| | | | | | $\ldots$ | | | | | | |
| $f(X)$ | 0 | 1 | 1 | 0 | 1 | 0 | $\ldots$ | | | | |
| $X \triangleleft f(X)$ | 0 | 1 | 0 | $\ldots$ | | | | | | | |

# MWC Stochasticity

von Mises-Wald-Church stochasticity, or MWC stochasticity, is similar to Church stochasticity, but we allow our selection functions to be partial. In this setting, rather than selecting a coin, we give a program that will run and possibly select it. We cannot go back and change our program or feed it information about later coins, but it does not have to halt before we move on to the next coin.

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ... | $f(1)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\ldots$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | $\ldots$ | $f(10)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(\emptyset)$ |
| | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(1)$ |
| | 1 | 0 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(10)$ |
| | 1 | 0 | 1 | ∘ | ∘ | ∘ | ∘ | ∘ | ∘ | … | $f(101)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(101)$ |
| | 1 | 0 | 1 | 1 | ○ | ○ | ○ | ○ | ○ | … | $f(1011)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(101)$ |
| | 1 | 0 | 1 | 1 | ○ | ○ | ○ | ○ | ○ | … | $f(1011)$ |
| | 1 | 0 | 1 | 1 | 0 | ○ | ○ | ○ | ○ | … | $f(10110)$ |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(101)$ |
| | 1 | 0 | 1 | 1 | ○ | ○ | ○ | ○ | ○ | … | $f(1011)$ |
| | 1 | 0 | 1 | 1 | 0 | ○ | ○ | ○ | ○ | … | $f(10110)$ |
| | | | | | … | | | | | | |
| $f(X)$ | 0 | 1 | 1 | 0 | 1 | 0 | … | | | | |

# Illustration for MWC Stochasticity

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| X | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(\emptyset)$ |
| | 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(1)$ |
| | 1 | 0 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(10)$ |
| | 1 | 0 | 1 | ○ | ○ | ○ | ○ | ○ | ○ | … | $f(101)$ |
| | 1 | 0 | 1 | 1 | ○ | ○ | ○ | ○ | ○ | … | $f(1011)$ |
| | 1 | 0 | 1 | 1 | 0 | ○ | ○ | ○ | ○ | … | $f(10110)$ |
| | | | | | … | | | | | | |
| $f(X)$ | 0 | 1 | 1 | 0 | 1 | 0 | … | | | | |
| $X \triangleleft f(X)$ | 0 | 1 | 0 | … | | | | | | | |

# Density vs. Stochasticity

Stochasticity and density are mathematically referring to the same idea, but they convey separate intentions. Stochasticity refers to the setting where both the parameter $r$ and the selection functions are fixed ahead of time. This is similar to the study of randomness where one fixes the measure first.

Density, on the other hand, refers to the setting where we fix the selection functions but allow the parameter to vary as in the case of intrinsic density. Here we are more concerned with how sets with different parameters interact with one another as opposed to studying the properties enjoyed by those of a specific parameter.

# Applications of `Into` and `Within`

We now turn our attention to studying how our noncomputable coding methods work in the realm of MWC and Church density. It turns out they work nearly identically to the intrinsic (injection) density case, but the classical operations are less well behaved. We shall work in MWC density, although the results will all hold for Church density as well.

As in the case of intrinsic density, $\mu_r$ (the $r$-Bernoulli measure) randoms for $r \in (0,1)$ give us sets of MWC-density $r$. 0 and 1 are not difficult to obtain, so as in the previous case we achieve the entire unit interval.

# The Within Operation

**Lemma**

*Suppose $C$ is computable and $A$ has MWC-density $\alpha$. Then $A \triangleleft C$ has MWC-density $\alpha$.*

Notice that this is exactly the same as the intrinsic density case.

# The Into Operation

**Theorem**

*Suppose that $A$ has MWC-density $\alpha$ relative to $B$ and $B$ has MWC-density $\beta$ relative to $A$. Then $B \triangleright A$ has MWC-density $\alpha\beta$.*

The proof idea is the same as the intrinsic density case, but the details are more complicated. It requires more relativization as MWC-density is allowed to change behavior based on previous bits. It is not known if this is tight.

# The Intersection

**Lemma**

*If $A$ has MWC-density $\alpha$ relative to $B$ and $B$ has MWC-density $\beta$ relative to $A$, then $A \cap B$ has MWC-density $\alpha\beta$.*

Note that in the intrinsic density case, this was an easy corollary of the previous theorem. The added relativization necessary in the MWC-density version causes problems with the proof. However, there is a separate proof that is not terribly complicated.

# The Join

`Into`, `within`, and intersection all mostly behaved as before. However, this trend will not continue.

## Lemma

*If $A$ does not have MWC-density* $1$*, then* $A \oplus A$ *does not have MWC-density.*

# Disjoint Union

**Lemma**

*There exist disjoint sets $A$ and $B$ of MWC-density $0$ such that $A \sqcup B$ does not have MWC-density.*

# Union

**Lemma**

*Suppose $A$ has MWC-density $\alpha$ relative to $B$ and $B$ has MWC-density $\beta$ relative to $A$. Then $A \cup B$ has MWC-density $\alpha + \beta - \alpha\beta$.*

Note that with these conditions, a previous lemma implies $A \cap B$ has MWC-density $\alpha\beta$. Thus if $A$ and $B$ are disjoint then one must have MWC-density 0.

# Special Disjoint Unions

There is a special form of disjoint union that we can rely on.

## Lemma

*Suppose that $A$ has MWC-density $\alpha$ relative to $B$ and $B$ has MWC-density $\beta$ relative to $A$. Then $A \sqcup (B \triangleright \overline{A})$ has MWC-density $\alpha + \beta(1 - \alpha) = \alpha + \beta - \alpha\beta$.*

This special form works for the construction we did in the intrinsic density case.

## Lemma

*If $q \in (0, 1)$ is a finite sum of $\frac{1}{2^{-1}}$'s, then any 1-Random computes a set of intrinsic density $q$.*

It is open whether we can extend this to infinite sums as we did for intrinsic density.

# Summary

The `into` and `within` operations allowed us to change intrinsic density and achieve everything in the unit interval. Their behavior in the MWC density case was much the same, although more relativization was required and the union did not work as well. We managed to recover finite unions of the form used for intrinsic density, but it remains open if we can use the infinite unions.