Definability and computability in uncountable structures

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Generic Muchnik Reducibility

Definition

A structure $\mathcal{M}$ is Muchnik reducible to $\mathcal{N}$ (written $\mathcal{M} \leq_w \mathcal{N}$) if every copy of $\mathcal{N}$ computes a copy of $\mathcal{M}$. (Note: We assume no uniformity.)

Definition (Schweber)

For (possibly uncountable structures) $\mathcal{M}$ and $\mathcal{N}$, we say that $\mathcal{M}$ is generic muchnik reducible to $\mathcal{N}$ ($\mathcal{M} \leq_w^* \mathcal{N}$) if for every (equivalently some) generic extension $V[G]$ of the set-theoretic universe V which makes $\mathcal{M}$ and $\mathcal{N}$ countable, we have $\mathcal{M} \leq_w \mathcal{N}$.

The fact that some or every can be used to describe the generic extension comes from Shoenfeld absoluteness.

Some familiar structures

$\mathcal{C}$ is the structure with universe 2^ω and language U_i so that $U_i(\sigma)$ if and only if $\sigma(i) = 1$.

Similarly, $\mathcal{B}$ is the structure with universe ω^ω and language $V_{i,j}$ so that $V_{i,j}(\sigma)$ if and only if $\sigma(i) = j$.

Note that for any countable structure $\mathcal{M}$, $\mathcal{M} <_w^* \mathcal{C}$.
To see this, fix any $\sigma \in \mathcal{C}$ which computes a copy of $\mathcal{M}$.
Then $\mathcal{M} \leq_w^* \{\sigma\} <_w^* \{\sigma'\} <_w^* \mathcal{C}$

Some familiar structures

Theorem (Knight-Montalban-Schweber, Igusa-Knight, Downey-Greenberg-Miller, Igusa-Knight-Schweber, A.-Knight-Kuyper-Miller-Soskova)

$$(\mathcal{C}, \oplus, ') \equiv \mathcal{BC}$$

$$\vee^{*}_{w}$$

$$(\mathcal{C}, \oplus) \equiv_w^* \mathcal{B} \equiv_w^* (\mathbb{R}, +, \cdot) \equiv_w^* (\mathbb{R}, +, <) \equiv_w^* (\mathbb{R}, +, \cdot, \{f_i\}_{i\in\omega})$$

$$\vee^{*}_{w}$$

$$\mathcal{C}$$

where the sequence $\{f_i\}_{i\in\omega}$ is any countable sequence of continuous functions on a power of $\mathbb{R}$.

$\mathcal{BC}$ is the “Borel complete” structure studied in A.-Knight-Kuyper-Miller-Soskova. For our purposes, we can define it to be $(\mathcal{C}, \oplus, ')$

Definition

A structure $\mathcal{M}$ is said to be Δ^0_2*-relatively-generically-categorical* if whenever A and B are two copies of $\mathcal{M}$ in $V[G]$, where $V[G]$ makes $\mathcal{M}$ countable, then there is a $\Delta^0_2(A \oplus B)$-isomorphism in $V[G]$ between them.

The fact that $\mathcal{C}$ and $\mathcal{B}$ are Δ^0_2-relatively-generically-categorical will mean that they are a good canvas for other structures to paint their definable sets.

Theorem

Both $\mathcal{C}$ and $\mathcal{B}$ are Δ^0_2-relatively-generically-categorical.

Proof.

It is computable in $A \oplus B$ to see that an element $a \in A$ has the same nth value as an element $b \in B$. Thus to check whether a has all the same values as b is $\Delta^0_2(A \oplus B)$. This gives the isomorphism. □

Starting towards definability

Definition

Let $\mathcal{M}$ be a countable structure. A set $X \subseteq \mathcal{M}^k$ is *relatively intrinsically* Σ^0_n *in* $\mathcal{M}$, written $\Sigma^*_n(\mathcal{M})$, if for every presentation $\mathcal{A}$ of $\mathcal{M}$, X is Σ^0_n relative to the atomic diagram of $\mathcal{A}$.

Theorem (Applied Ash–Knight–Manasse–Slaman–Chisholm)

Let $V[G]$ be a generic extension and $X \in V[G]$ be a set that is $\Sigma^*_n(\mathcal{M})$ in the sense of $V[G]$. Then there is a Σ^c_n formula in the sense of V that defines X in V. In particular, $X \in V$.

Proof.

The classic Ash–Knight–Manasse–Slaman–Chisholm theorem shows that there is a Σ^c_n formula $\varphi(x, \vec{a})$ in $V[G]$ that defines X (with finitely many parameters from $\mathcal{M}$) in $V[G]$. But computability is absolute, so φ is Σ^c_n and in V. Similarly, satisfaction of φ on $\mathcal{M}$ is absolute, so the fact that X is defined by φ is absolute. □

Complexity Profiles

Definition

Let $\mathcal{M}$ be a structure with a copy of $\mathcal{A}$ in it (i.e. quantifier-free defined in $\mathcal{M}$). Then we say a set $X \subseteq \mathcal{A}$ is in $\mathcal{A}\Sigma_i^{\mathcal{M}}$ if X (as a subset of $\mathcal{A}$) is relatively intrinsically Σ_i in $\mathcal{M}$.

Notation overlap: Note that $\Sigma_n^*(\mathcal{A}) = \mathcal{A}\Sigma_n^{\mathcal{A}}$.

Theorem

If $\mathcal{A}$ is Δ_2^0-relatively-generically-categorical, then which copy of $\mathcal{A}$ is used in the definition of $\mathcal{A}\Sigma_i^{\mathcal{M}}$ for $i \geq 2$ doesn't matter. This lets us define the collection $\mathcal{A}\Sigma_i^{\mathcal{M}}$ for any $\mathcal{M} \geq_w^* \mathcal{A}$.

Formally, for any $\mathcal{M} \geq_w^* \mathcal{A}$, we replace $\mathcal{M}$ by $\mathcal{M} \sqcup \mathcal{A}$ and look at that particular chosen copy of $\mathcal{A}$.

Further, if $\mathcal{M} \leq_w^* \mathcal{N}$, then $\mathcal{A}\Sigma_i^{\mathcal{M}} \subseteq \mathcal{A}\Sigma_i^{\mathcal{N}}$.

The $\mathcal{A}$-*Complexity profile of* $\mathcal{M}$ is the sequence $(\mathcal{A}\Sigma_i^{\mathcal{M}})_{i\geq 2}$.

Some topology notation

Definition

For a Polish space X (we'll apply this to $\mathcal{C}$ and $\mathcal{B}$):

- The *Borel sets* of X are the sets in the σ-algebra generated by the open sets of X.
- $\mathbf{\Sigma^0_1}$ is the collection of open subsets of X.
- $\mathbf{\Pi^0_k}$ is the collection of complements of $\mathbf{\Sigma^0_k}$ sets.
- $\mathbf{\Sigma^0_{k+1}}$ is the collection of countable unions of $\mathbf{\Pi^0_k}$ sets.
- $\mathbf{\Delta^0_k}$ is $\mathbf{\Sigma^0_k} \cap \mathbf{\Pi^0_k}$.

- A set $A \subseteq X$ is $\mathbf{\Sigma^1_1}$ if for some Polish Y and Borel $B \subseteq X \times Y$, A is the projection of B on X.
- $\mathbf{\Pi^1_k}$ is the collection of complements of $\mathbf{\Sigma^1_k}$ sets.
- A set $A \subseteq X$ is $\mathbf{\Sigma^1_{k+1}}$ if for some Polish Y and $\mathbf{\Pi^1_k}$ set $B \subseteq X \times Y$, A is the projection of B on X.
- $\mathbf{\Delta^1_k}$ is $\mathbf{\Sigma^1_k} \cap \mathbf{\Pi^1_k}$.
- A set is *projective* if it is in $\bigcup_{k\in\omega} \mathbf{\Sigma^1_k}$.

The complexity profiles

Theorem

1. The $\mathcal{C}$-complexity profile of $\mathcal{C}$ is given by $\mathcal{C}\Sigma_2^{\mathcal{C}} = \mathbf{\Sigma_2^0}$, and $\mathcal{C}\Sigma_i^{\mathcal{C}} = \mathbf{\Sigma_{i-2}^1}$ for $i \geq 3$.
2. The $\mathcal{B}$-complexity profile of $\mathcal{B}$ is given by $\mathcal{B}\Sigma_i^{\mathcal{B}} = \mathbf{\Sigma_{i-1}^1}$.
3. The $\mathcal{C}$-complexity profile of $\mathcal{B}$ is given by $\mathcal{C}\Sigma_i^{\mathcal{B}} = \mathbf{\Sigma_{i-1}^1}$.
4. The $\mathcal{B}$-complexity profile of $\mathcal{BC}$ is given by $\mathcal{B}\Sigma_i^{\mathcal{BC}} = \mathbf{\Sigma_i^1}$.
5. The $\mathcal{C}$-complexity profile of $\mathcal{BC}$ is given by $\mathcal{C}\Sigma_i^{\mathcal{BC}} = \mathbf{\Sigma_i^1}$.

Proof.

Ash–Knight–Menasse–Slaman–Chisholm + some care. □

Corollary (Already known, but this gives a clear reason in terms of definability)

$\mathcal{C} <_w^* \mathcal{B} <_w^* \mathcal{BC}$

Expansions of $\mathcal{C}$ or $\mathcal{B}$ by unary predicates

Theorem

Let $\mathcal{A}$ be either $\mathcal{C}$ or $\mathcal{B}$. Let $(S_i)_{i\in\omega}$ be a countable sequence of unary relations. Let $F\colon \mathcal{A} \to \mathcal{A}$ be given by $F(x) = y$ where $y(n) = 1$ if $x \in S_n$ and $y(n) = 0$ if $x \notin S_n$ (so F actually has range in $\mathcal{C} \subseteq \mathcal{A}$). Then $(\mathcal{A}, (S_i)_{i\in\omega}) \equiv^*_w \mathcal{A}$ if and only if the graph of F is in $\Delta^*_2(\mathcal{A})$.

Proof.

Suppose that $(\mathcal{A}, (S_i)_{i\in\omega}) \equiv^*_w \mathcal{A}$. Then $\mathcal{A}\Delta_2^{\mathcal{A}} = \mathcal{A}\Delta_2^{(\mathcal{A},(S_i)_{i\in\omega})}$, and F is in the latter.
Supposing $F \in \mathcal{A}\Delta_2^{\mathcal{A}}$, we do a finite injury argument giving ourselves a single element from $\mathcal{A}$ which tells us, for each σ and boolean combination of the S_i whether there is some $\tau \in \mathcal{A}$ extending σ in the boolean combination, and if so, it specifies one. □

Expansions of $\mathcal{C}$ or $\mathcal{B}$ by closed relations

Definition

A formula is *positive existential, also written* $\exists^+$, if it is in the closure of atomic formulas by the operations $\wedge, \vee, \exists v$.
Let $\mathcal{A} \in \{\mathcal{C}, \mathcal{B}\}$ and let $\mathcal{M}$ be an expansion of $\mathcal{A}$. We define $\mathrm{SMA}_k(\mathcal{M}) \subseteq \mathcal{A}^{k+1}$ to be the relation defined by: $\mathrm{SMA}_k(\bar{x}, y)$ holds if $|\bar{x}| = k$ and the positive existential type of $\bar{x}$ in $\mathcal{M}$ equals y. That is, y defines the characteristic function of the positive existential type of $\bar{x}$, identifying formulas with numbers via their Gödel codes.
We define $\mathrm{SMA}(\mathcal{M})$ to be a unary relation on $\mathcal{A}$ given by $\mathrm{SMA}(z)$ holds if $z = n \frown (x_1 \oplus \cdots \oplus x_n \oplus y)$ and $\mathrm{SMA}_n(\bar{x}, y)$.

Lemma

Let $\mathcal{M}$ be an expansion of $\mathcal{C}$ or $\mathcal{B}$ by countably many closed relations. Then $\mathrm{SMA}(\mathcal{M}) \in \Delta^*_2(\mathcal{M})$.

Expansions of $\mathcal{C}$ or $\mathcal{B}$ by closed relations

Theorem

Let $\mathcal{A}$ be $\mathcal{C}$ or $\mathcal{B}$ and let $\mathcal{H} \geq_w^* \mathcal{A}$. Let $\mathcal{M}$ be an expansion of $\mathcal{A}$ by countably many closed relations. Then $\mathcal{M} \leq_w^* \mathcal{H}$ if and only if $\mathrm{SMA}(\mathcal{M}) \in \mathcal{A}\Delta_2^{\mathcal{H}}$.

Proof.

If $\mathcal{M} \leq_w^* \mathcal{H}$, then $\mathrm{SMA}(\mathcal{M}) \in \mathcal{A}\Delta_2^{\mathcal{M}} \subseteq \mathcal{A}\Delta_2^{\mathcal{H}}$. The other direction is a subtle "pull-down" argument where SMA is exactly what we need to have guessed in order to guide our recovery from injury. □

Once again, keeping the same generic muchnik degree is the same as keeping the same complexity profile. The collection of definable sets seems to be a fine enough notion.

Expansions of $\mathcal{C}$ that must be above $\mathcal{B}$

Observation

$(\mathcal{C}, P)$ where P is a predicate for the "rationals" (i.e. the elements of $\mathcal{C}$ which have only finitely many 1s) is generic Muchnik equivalent to $\mathcal{B}$.

Proof.

For each $x \notin P$, let $\pi(x) \in \mathcal{B}$ be the element describing the distances between consecutive 1s in x. Then π is a computable bijection between $\mathcal{C} \setminus P$ and $\mathcal{B}$. □

A fancier version of this same idea is:

Theorem

Suppose $\mathcal{M} \geq_w^* \mathcal{C}$ and $A \subseteq \mathcal{C}\Delta_2^{\mathcal{M}}$ is countable, $P \subseteq \mathcal{C}$ is perfect, and $A \cap P$ is dense in P. Then $\mathcal{M} \geq_w^* \mathcal{B}$.

Δ^*_2-sets which are not $\mathbf{\Delta^0_2}$ get us up to $\mathcal{B}$

Lemma (Hurewicz)

If $R \subseteq \mathcal{C}$ is Borel but not $\mathbf{\Delta^0_2}$, then there is a perfect set $P \subseteq \mathcal{C}$ such that either $P \cap R$ or $P \setminus R$ is countable and dense in P.

Corollary

If $\mathcal{M} \leq^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ and $\Delta^*_2(\mathcal{M}) \neq \Delta^*_2(\mathcal{C})$, then $\mathcal{M} \equiv^*_w \mathcal{B}$.

Proof.

Let $X \in \Delta^*_2(\mathcal{M}) \setminus \Delta^*_2(\mathcal{C})$. Since $\mathcal{M} \leq^*_w \mathcal{B}$, X is in $\mathcal{C}\Delta^{\mathcal{B}}_2$. Then, X is Borel, but not $\mathbf{\Delta^0_2}$. Hurewicz and the previous Theorem give that $\mathcal{M} \geq^*_w \mathcal{B}$. □

Theorem

Let $(U_i)_{i\in\omega}$ be a countable sequence of unary relations on $\mathcal{C}$. Then $\mathcal{M} = (\mathcal{C}, (U_i)_{i\in\omega})$ cannot have generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$.

Proof.

Suppose $\mathcal{M} \leq_w^* \mathcal{B}$. Then since each $U_i \in \mathcal{C}\Delta_2^{\mathcal{M}}$, we either have $\mathcal{M} \geq_w^* \mathcal{B}$ or they are all in $\Delta_2^*(\mathcal{C})$, i.e., they are all $\boldsymbol{\Delta_2^0}$. But then the F which codes up the unary predicates is Borel and in $\mathcal{C}\Delta_2^{\mathcal{M}}$. So either F is $\boldsymbol{\Delta_2^0}$, in which case $\mathcal{M} \leq_w^* \mathcal{C}$ or it is not, in which case, $\mathcal{M} \geq_w^* \mathcal{B}$. □

Dichotomy theorems for expansions of $\mathcal{C}$ Part II

Theorem

Suppose that $\mathcal{M}$ is an expansion of $\mathcal{C}$ by countably many closed relations. Then either $\mathcal{M} \equiv^*_w \mathcal{C}$ or $\mathcal{M} \geq^*_w \mathcal{B}$.

Proof.

Either SMA($\mathcal{M}$) is $\mathbf{\Delta^0_2}$, in which case $\mathcal{M} \leq^*_w \mathcal{C}$ or it is not, in which case $\mathcal{M} \geq^*_w \mathcal{B}$, because SMA($\mathcal{M}$) $\in \Delta^*_2(\mathcal{M})$ and is Borel. □

Expansions of $\mathcal{B}$ that must be above $\mathcal{BC}$

Lemma

Let $\mathcal{M}$ be an expansion of $\mathcal{B}$. Suppose that there is a set $Y \in \Delta_2^*(\mathcal{M})$ that is $\boldsymbol{\Sigma_1^1}$-hard under continuous reduction. Then $\mathcal{M} \geq_w^* \mathcal{BC}$.

We will assume enough Wadge determinacy so that for any set in $\Delta_2^*(\mathcal{M})$, either X is Borel or X (or its complement) is $\boldsymbol{\Sigma_1^1}$-hard under continuous reduction. For example:

Theorem (Projective Wadge-determinacy)

Let $\mathcal{M} = (\mathcal{B}, X)$ where X is projective. Suppose $\Delta_2^*(\mathcal{M}) \neq \Delta_2^*(\mathcal{B})$. Then $\mathcal{M} \geq_w^* \mathcal{BC}$.

Proof.

Let Y be in $\Delta_2^*(\mathcal{M}) \setminus \Delta_2^*(\mathcal{B})$. Then Y is projective but not Borel. So either Y or its complement is $\boldsymbol{\Sigma_1^1}$-hard under continuous reduction, so $\mathcal{M} \geq_w^* \mathcal{BC}$. $\square$

Theorem ($\mathbf{\Delta^1_2}$-Wadge determinacy)

Let $(U_i)_{i\in\omega}$ be a countable sequence of unary relations on $\mathcal{B}$. Then $\mathcal{M} = (\mathcal{B}, (U_i)_{i\in\omega})$ cannot have generic Muchnik degree strictly between $\mathcal{B}$ and $\mathcal{BC}$.

Proof.

Suppose that $\mathcal{M} \leq^*_w \mathcal{BC}$, then each U_i is $\mathbf{\Delta^1_2}$ ($=\mathcal{B}\Delta_2^{\mathcal{BC}}$). If any are not Borel, then $\mathbf{\Delta^1_2}$-Wadge determinacy says that it or its complement is $\mathbf{\Sigma^1_1}$-hard under continuous reduction. Then $\mathcal{M} \geq^*_w \mathcal{BC}$. So we can suppose they are all Borel. Then the function F which gathers them up is $\mathbf{\Delta^1_2}$.
Either F is Borel($=\Delta^*_2(\mathcal{B})$), in which case $\mathcal{M} \leq^*_w \mathcal{B}$ or it is not, in which case, $\mathcal{M} \geq^*_w \mathcal{BC}$. □

Dichotomy theorems for expansions of $\mathcal{B}$ Part II

Theorem ($\mathbf{\Delta^1_2}$-Wadge determinacy)

Suppose that $\mathcal{M}$ is an expansion of $\mathcal{B}$ by countably many closed relations. Then either $\mathcal{M} \equiv^*_w \mathcal{B}$ or $\mathcal{M} \geq^*_w \mathcal{BC}$.

Proof.

Either SMA($\mathcal{M}$) is $\Delta^*_2(\mathcal{B})$ (i.e. Borel), in which case $\mathcal{M} \leq^*_w \mathcal{B}$ or it is not, in which case, SMA($\mathcal{M}$) $\in \Delta^*_2(\mathcal{M}) \subseteq \mathbf{\Delta^1_2}$, so $\mathcal{M} \geq^*_w \mathcal{BC}$ by the same Wadge determinacy + $\mathbf{\Sigma^1_1}$-hardness argument. □

Building intermediate degrees

How can we build some degree between $\mathcal{C}$ and $\mathcal{B}$?
The idea is to build some structure $\mathcal{C} \leq_w^* \mathcal{M} \leq_w^* \mathcal{B}$ so that $\mathcal{C}\Delta_2^{\mathcal{M}} = \mathcal{C}\Delta_2^{\mathcal{C}}$, but $\mathcal{C}\Delta_k^{\mathcal{M}} \supsetneq \mathcal{C}\Delta_k^{\mathcal{C}}$ for some k. The former ensures that $\mathcal{M} \not\geq_w^* \mathcal{B}$, and the latter ensures that $\mathcal{C} \not\geq_w^* \mathcal{M}$.
The strategy uses the following theorem:

Theorem (Knight)

If a linear order has a jump degree, then it must be $0'$.

We extend this to:

Theorem

Let $\mathcal{M} \geq_w^* \mathcal{C}$ and L be a linear order. Then $\mathcal{C}\Delta_2^{\mathcal{M}} = \mathcal{C}\Delta_2^{\mathcal{M}\sqcup L}$.

So a structure of the form $\mathcal{C} \sqcup L$ necessarily satisfies the first condition $\mathcal{C}\Delta_2^{\mathcal{M}} = \mathcal{C}\Delta_2^{\mathcal{C}}$. With some coding, we can use this idea to build degrees strictly between $\mathcal{C}$ and $\mathcal{B}$, and also strictly between $\mathcal{B}$ and $\mathcal{BC}$.

Conclusions

The complexity profiles capture which sets are definable at various levels of the $\mathcal{L}^c_{\omega_1,\omega}$-hierarchy. For "natural" structures, these correspond to topologically meaningful classes.

Complexity profiles are subtle enough to sense (at least in these examples) when structures have the same or different generic Muchnik degree, and even to provide dichotomy theorems.

They are also sensitive enough to use to build intermediate degrees.

Overall, complexity profiles are a great tool in understanding generic Muchnik degrees. And despite my innate set-theory aversion, I think that the generic Muchnik degrees are the (or at least 'a') correct way to extend the ideas of computability into the uncountable setting.

The end

Thank you!