Fickleness and Bounding Lattices in $\mathcal{R}_T$

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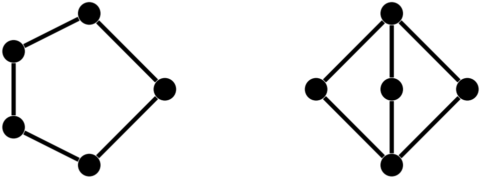
Motivation

Understand the relation between the *fickleness* of a recursively enumerable (r.e.) Turing degree $\mathbf{d} \in \mathcal{R}_\mathrm{T}$ and its ability to bound a given finite lattice $(L, \vee, \wedge)$.

Bounding Distributive Lattices in $\mathcal{R}_T$

Lattices can be distributive or non-distributive. Distributive lattices are those that do not contain a copy of N_5 or 1-3-1 as sublattices (Birkhoff).



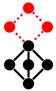
N_5 1-3-1

Theorem (Lerman; Lachlan 1972; Thomason 1971)

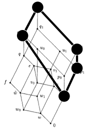
Distributive lattices can be bounded below any $\mathbf{d} \in \mathcal{R}_T - \{0\}$.

Bounding Non-Distributive Lattices in $\mathcal{R}_T$

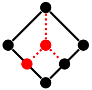
S_8



L_{20}



L_7



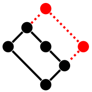


Fig: Some non-distributive lattices. They must contain N_5 or 1-3-1.

Let $\mathbf{d} \in \mathcal{R}_T - \{0\}$.

Theorem (Lachlan and Soare 1980; Lempp and Lerman 1997; Downey, Greenberg, and Weber 2007; Ambos-Spies and Losert 2019; Downey and Greenberg 2015)

d *bounds N_5 (Folklore).*

d *cannot bound S_8 or L_{20} (LS80;LL97).*

d *bounds L_7 iff its "fickleness $> \omega$" (DGW07;AL19).*

d *bounds 1-3-1 iff its "fickleness $\geq \omega^\omega$" (DG15).*

Fickleness of $\mathbf{d} \in \mathcal{R}_\mathsf{T}$

Let $\mathbf{d} \in \mathcal{R}_\mathsf{T}$, $\alpha \leq \epsilon_0 := \sup\left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\right\}$.

Definition (Downey and Greenberg 2015)

A set is α*-computably approximable (*α*-c.a.)* if it "changes its mind $\leq \alpha$-times". E.g. *n*-r.e. sets are *n*-c.a..
d is *totally* α*-c.a.* ($\mathbf{d} \in \mathrm{T}(\alpha)$, or **d**'s *fickleness* $\leq \alpha$) if every $A \in \mathbf{d}$ is α-c.a..
d is *properly* $\mathrm{T}(\alpha)$ ($\mathbf{d} \in \mathrm{pT}(\alpha)$, or **d**'s *fickleness* $= \alpha$) if $\mathbf{d} \in \mathrm{T}(\alpha)$ and $\mathbf{d} \notin \mathrm{T}(\beta)\ \forall \beta < \alpha$.

Fickleness Hierarchy

Theorem (Downey and Greenberg 2015)

For every $\alpha \leq \epsilon_0$ there exists $\mathbf{d} \in \mathrm{pT}(\omega^\alpha)$.
If $\mathbf{d} \in \mathrm{T}(\beta)$ *and* $\omega^\alpha \leq \beta$ *is the largest power of* ω *below* β, *then* $\mathbf{d} \in \mathrm{T}(\omega^\alpha)$.
Every $\mathbf{d} \in \mathrm{T}(\omega^\alpha)$ *is* low_2.

Lemma

For every $\alpha \leq \epsilon_0$ there exists low and nonlow $\mathbf{d} \in \mathrm{pT}(\omega^\alpha)$.

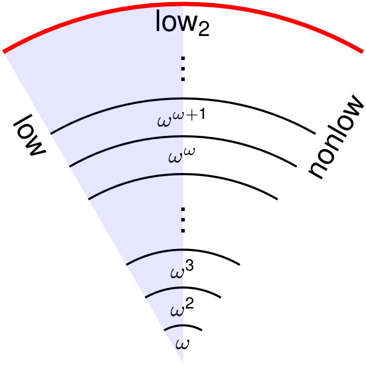
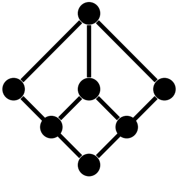


Figure: Fickleness hierarchy is low_2, independent from nonlowness, and collapses to powers of ω.

Towards Characterizing $> \omega^2$-Fickleness

Open Question (Downey and Greenberg 2015)

We saw that L_7 (1-3-1) characterized $> \omega$ ($\geq \omega^\omega$) -fickleness.
Is there a lattice that characterizes $> \omega^2$-fickleness?

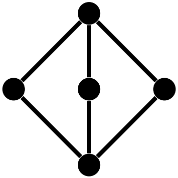


L_7

$> \omega$-fickleness

?

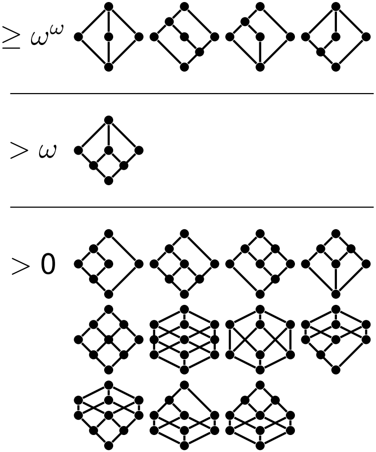
$> \omega^2$-fickleness



1-3-1

$\geq \omega^\omega$-fickleness

3 Independent Elements Lattices Do Not Characterize $> \omega^2$-Fickleness

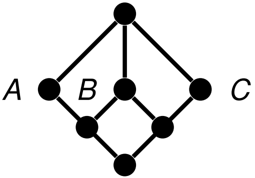


Consider lattices L like L_7 and 1-3-1 with no more than 3 independent elements A, B, C, and every element in L is either the join or meet of elements in $\{A, B, C\}$.

Theorem

Each such lattice either characterizes > 0, $> \omega$, or $\geq \omega^\omega$-fickleness.

One Meet Demands $> \omega$-Fickleness



(Downey, Greenberg, and Weber 2007; Ambos-Spies and Losert 2019)
Construct r.e. A, B, C, Δ_A, Δ_C satisfying

$$\begin{aligned} J_A &: A = \Delta_A(B, C), \\ J_C &: C = \Delta_C(A, B), \\ D_\Psi &: A \neq \Psi(B), \\ M_\Phi &: \Phi_0(A) = \Phi_1(C) = W \implies W \leq 0. \end{aligned}$$

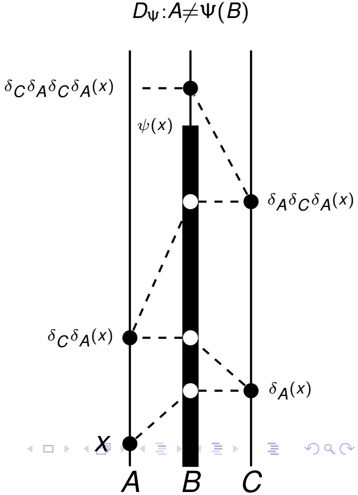
J_A-strategy: To put x into A, first put $\delta_A(x)$ into B or C.
J_C-strategy: To put x into C, first put $\delta_C(x)$ into A or B.

One Meet Demands $> \omega$-Fickleness

D-strategy: Pick x and wait for x to be *realized* ($\Psi(x) = 0$). Restrain $B \upharpoonright \psi(x)$. Want to put x into A, but J_A requires $\delta_A(x)$ be put into B or C first. Restraint on B forces us to target C. J_C requires $\delta_C(\delta_A(x))$ be put into A or B first. Restraint on B forces us to target A. Repeat till we can target B when

$$\underbrace{\delta_C(\delta_A(\ldots\delta_C(\delta_A(}_{n \text{ alternations}} x))\ldots) > \psi(x).$$

We get an *ac*-trace $x, \delta_A(x), \delta_C\delta_A(x), \ldots$ of length $n < \omega$ that needs to be enumerated into A and C in reverse before x finally enters A.



One Meet Demands $> \omega$-Fickleness

$$M_\Phi : \Phi_0(A) = \Phi_1(C) = W \implies W \leq 0.$$

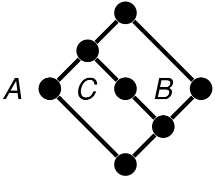
M-strategy: Wait for equality $\Phi_0(y) = \Phi_1(y)$. Always restrain $A \upharpoonright \phi_0(y)$ or $C \upharpoonright \phi_1(y)$ to prevent injuring computations on A and C sides simultaneously.

D versus M: D needs to enumerate an *ac*-trace of length $n < \omega$. M disallows the entire trace from being enumerated simultaneously, so D needs n permissions to be satisfied.

Construction can be viewed as a pinball machine, where an *ac*-trace is represented by *ac*-balls, and where M is represented as an *AC*-gate that opens and closes infinitely often, allowing only one ball to pass through each time.

$M : A \wedge C = 0$ ——— *acacac*

Two Equal Meets Demand $\geq \omega^\omega$-Fickleness



(Downey and Greenberg 2015)
Construct r.e. A, B, C, Δ_A, Δ_C satisfying

$$J_A : A = \Delta_A(B, C),$$
$$J_C : C = \Delta_C(A, B),$$
$$D_\Psi : A \neq \Psi(B),$$
$$M_{AC\Phi} : \Phi_0(A) = \Phi_1(C) = W \implies W \leq 0,$$
$$M_{AB\Phi} : \Phi_0(A) = \Phi_1(B) = W \implies W \leq 0.$$

Two Equal Meets Demand $\geq \omega^\omega$-Fickleness

The new M_{AB} requirement introduces *AB*-gates for *ac*-traces to pass through.

acacac

$A \wedge C = 0$

$A \wedge B = 0$

$A \wedge C = 0$

$A \wedge B = 0$

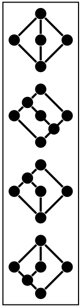
To pass 2 (k) alternations of *AC* and *AB*-gates, the trace demands $\geq \omega^2$ ($\geq \omega^k$) permissions. Therefore with just one more meet requirement, fickleness demanded increases from $> \omega$ to $\geq \omega^\omega$.

Alternative Conditions that Demand $\geq \omega^\omega$-Fickleness

Open Question

Besides having two equal meets and relevant join requirements, are there other sets of conditions a lattice could satisfy to demand $\geq \omega^\omega$-fickleness?
In particular, can we find a 4 independent element lattice L at the $\geq \omega^\omega$ level that does not already contain a copy of any of these $\geq \omega^\omega$ latices?

$\geq \omega^\omega$



Alternative Conditions that Demand $\geq \omega^\omega$-Fickleness

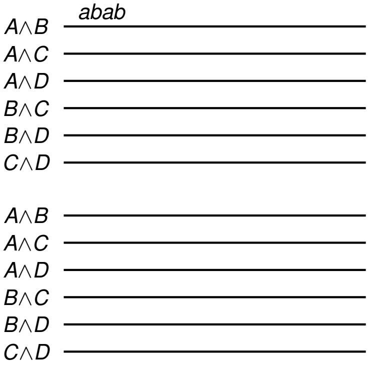
Consider a lattice L with 4 independent elements A, B, C, D, satisfying

$$
\begin{aligned}
A &\leq B + C + D, & A \wedge B &= 0, \\
B &\leq A + C + D, & A \wedge C &= 0, \\
C &\leq A + B + D, & A \wedge D &= 0, \\
D &\leq A + B + C, & B \wedge C &= 0, \\
& & B \wedge D &= 0, \\
& & C \wedge D &= 0.
\end{aligned}
$$

Lemma

Any L satisfying the above demands $\geq \omega^\omega$-fickleness.

The pinball construction hints at the $\geq \omega^\omega$-fickleness demanded:

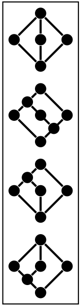


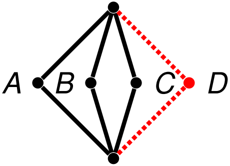
Alternative Conditions that Demand $\geq \omega^\omega$-Fickleness

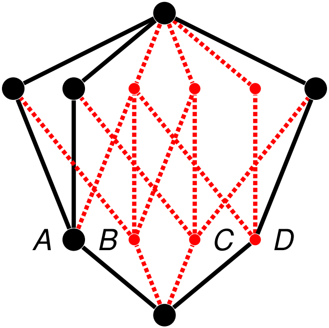
Conjecture

Every lattice L satisfying the previous conditions already contains a copy of a 3 independent elements lattice that demands $\geq \omega^\omega$-fickleness.

$\geq \omega^\omega$







Infinite Semilattice

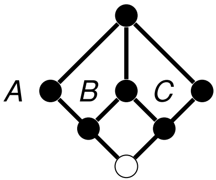
Open Question

Are there infinite semilattices that characterize $\geq \omega^2$-fickleness?

Consider the infinite upper semilattice obtained by removing the meet from L_7, i.e. $A \cap C$ does not exist.

Theorem

L_7 without meet characterizes $> \omega$-fickleness.



$$J_A : A = \Delta_A(B, C),$$
$$J_C : C = \Delta_C(A, B),$$
$$D_\Psi : A \neq \Psi(B),$$
$$M'_\Phi : \Phi_0(A) = \Phi_1(C) = W \implies (\exists \kappa) W = \kappa(B),$$
$$R : A \wedge C \text{ does not exist.}$$

One Non-meet Demands $> \omega$-Fickleness

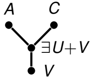
$$M'_\Phi : \Phi_0(A) = \Phi_1(C) = W \implies (\exists \kappa) W = \kappa(B)$$

M'-strategy: Wait for $\Phi_0(A, y) = \Phi_1(C, y)$. Pick large use $k(y)$. Allow simultaneous injury on A and C sides only if some $b \leq k(y)$ enters B at the same time.

D vs M': D wants to enumerate an *ac*-trace. To minimize demanded fickleness we are tempted to enumerate the entire trace simultaneously. But that requires us to put some b into B. M' needs to know this b early, possibly picking b before D is realized. But then D will be unrealized when b enters B. So we cannot avoid enumerating the *ac*-trace one element at a time and demanding $> \omega$-fickleness.

One Non-meet Demands $> \omega$-Fickleness

$$R_{V\Gamma\Xi} : V \not\geq A \;\&\; \begin{array}{l} \Gamma_0(A) = \Gamma_1(C) = V \\ \Theta_0(A) = \Theta_1(C) = U \end{array} \;\; \begin{array}{l} \implies (\exists\Theta, U) \\ \& \; U \neq \Xi(V). \end{array}$$



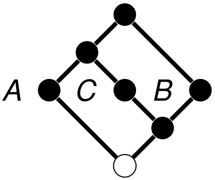
R-strategy (Ambos-Spies 1984): Pick large $x, \theta_0(x), \theta_1(x)$. Wait for x to be realized ($\Xi(x) = 0$). Wait for $\theta_0(x)$ to be lifted above the use for realization, which must occur if $V \not\geq A$. Simultaneously put $\theta_0(x)$ into A, $\theta_1(x)$ into C, x into U. Restrain A to prevent unrealization.

R vs M': R simultaneously injures A and C computations. M' allows that if some b also enters B. M' needs b to be picked early, sometimes before R is realized. This is alright because R never restrains B. All enumerations are done simultaneously, so 1 permission is enough.

Two Equal Non-meets Demand $\geq \omega^\omega$-Fickleness

Theorem

Consider the lattice L shown below, which is the same as the earlier 2-meet lattice after ensuring that $A \wedge B$ and $A \wedge C$ do not exist. L characterizes $\geq \omega^\omega$-fickleness.



$$
\begin{aligned}
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J_C &: C = \Delta_C(A, B), \\
D_\Psi &: A \neq \Psi(B), \\
M'_{AC\Phi} &: \Phi_0(A) = \Phi_1(C) = W \implies (\exists \kappa) W = \kappa(B), \\
M'_{AB\Phi} &: \Phi_0(A) = \Phi_1(B) = W \implies (\exists \kappa) W = \kappa(C), \\
R &: A \wedge C \text{ does not exist.}
\end{aligned}
$$

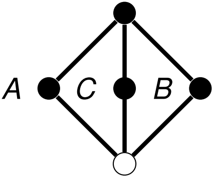
Two Equal Non-meets Demand $\geq \omega^\omega$-Fickleness

D vs M'_{AC}, M'_{AB}: Like before, even though M' allows simultaneous injury, we cannot reduce the demanded fickleness because we might unrealize D.

R vs M'_{AC}, M'_{AB}: Like before, R needs to put some a into A and c into C simultaneously. M'_{AC} allows this because R can pick some b early enough to be put into B. But by M'_{AB}, the a, b enumerations forces R to pick some c' early enough to be put into C. We can choose $c' = c$ since R does not impose a restraint on C. All enumerations are done simultaneously, so fickleness of 1 is sufficient.

Three Equal Non-meets

What if we add the final type of M' requirement?



1-3-1 without meet

$$
\begin{aligned}
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M'_{AB\Phi} &: \Phi_0(A) = \Phi_1(B) = W \implies (\exists \kappa) W = \kappa(C),\\
M'_{BC\Phi} &: \Phi_0(B) = \Phi_1(C) = W \implies (\exists \kappa) W = \kappa(A),\\
R &: A \wedge C \text{ does not exist.}
\end{aligned}
$$

Three Equal Non-meets

R vs M'_{AB}, M'_{AC}, M'_{BC}: R cannot help but injure the A, B, C-gates simultaneously via a, b, c traces. M'_{AB} (M'_{AC}) allowed the AB-injury (AC) because c (b) could be chosen early enough. Likewise, M'_{BC} will allow the BC-injury if a can be chosen early enough. But we cannot choose a early if we want to avoid unrealizing R.

Conjecture

1-3-1 without meet cannot be bounded in the r.e. degrees.