The structure of Weihrauch degrees - what we know and what we don't know

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MidWest Computability Seminar 2021

2017: The survey

Vasco Brattka, Guido Gherardi & Arno Pauly:
Weihrauch Complexity in Computable Analysis.
arXiv 1707.03202

And an update

What happened since? What are some interesting open questions?

Arno Pauly:
An update on Weihrauch complexity, and some open questions.
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A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.

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Represented spaces and computability

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbf{2}^{\mathbb{N}} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is a realizer of $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \mathrm{dom}(f\delta_X)$.

$$\begin{array}{ccc} \mathbf{2}^{\mathbb{N}} & \xrightarrow{F} & \mathbf{2}^{\mathbb{N}} \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

Definition

$f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

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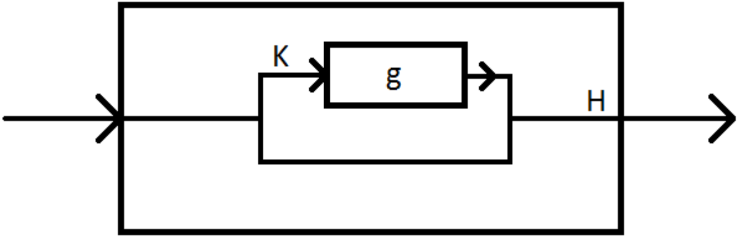
Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $H\langle \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, GK\rangle$ is a realizer of f for every realizer G of g. $\mathfrak{W}$ denotes the Weihrauch degrees.



Weihrauch reducibility on Baire space

Proposition

For $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ we that $f \leq_W g$ iff there are computable $H, K \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $K : \mathrm{dom}(f) \to \mathrm{dom}(g)$ such that $H(\langle p, q\rangle) \in f(p)$ for all $q \in g(K(p))$.

What people are working on

- Most work on Weihrauch degrees is about classifying specific theorems.
- Then there is work on creating a “scaffolding” of stuff like closed choice principles.
- But only a few papers on the structure of the Weihrauch degrees.
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Distributive lattice

Theorem (Brattka & Gherardi; Pauly)

The Weihrauch degrees form a distributive lattice;

- *with join* $\sqcup$ *induced by* $(f \sqcup g) :\subseteq \mathbf{X} + \mathbf{U} \rightrightarrows \mathbf{Y} + \mathbf{U}$, $(f \sqcup g)(0, x) = (0, f(x))$ *and* $(f \sqcup g)(1, y) = (1, g(y))$,
- *and with meet* $\sqcap$ *induced by* $(f \sqcap g) :\subseteq \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} + \mathbf{V}$, $(f \sqcap g)(x, y) = (0 \times f(x)) \cup (1 \times g(y))$.

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Special degrees

- The least element is 0, the trivially true principle without instances.
- With 1 we denote the degree of $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}$ comprised of all computable problems with a computable instance.
- And $\emptyset$ is the top element (which is probably fake).

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Incompleteness

Theorem (Higuchi & Pauly)

No non-trivial suprema exist in the Weihrauch lattice, meaning either $\sqcup_{i\in\mathbb{N}} f_i$ does not exist, or there is some $N \in \mathbb{N}$ with $\sqcup_{i\in\mathbb{N}} f_i = \sqcup_{i\leq N} f_i$.

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Heyting algebra?

Question (Brattka & Gherardi)

Is the Weihrauch lattice a Brouwer algebra, i.e. does

$$\inf_{\leq_W}\{h \mid g \leq_W f \sqcup h\}$$

exist for all f, g?

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The Weihrauch lattice is neither a Brouwer not a Heyting algebra.

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Medvedev degrees

Definition (Medvedev reducibility)
For $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, $A \leq_M B$ iff $\exists F : B \to A$, F computable. Let $\mathfrak{M}$ denote the Medvedev degrees.

Theorem (Brattka & Gherardi)
$A \mapsto c_A$, where $c_A(p) = A$, is a meet-semilattice embedding of $\mathfrak{M}$ into $\mathfrak{W}$.

Theorem (Higuchi & Pauly)
$A \mapsto d_A$, where $d_A : A \to \{0\}$, is a lattice embedding of $\mathfrak{M}^{\mathrm{op}}$ into $\mathfrak{W}$. In fact, it is an isomorphism between $\mathfrak{M}^{\mathrm{op}}$ and $\{f \in \mathfrak{W} \mid 0 <_W f \leq_W 1\}$.

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Is there a lattice-embedding of $\mathfrak{M}$ into $\mathfrak{W}$?

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Many-one degrees

Definition (Many-one reductions)

For $A, B \subseteq \mathbb{N}$, let $A \leq_m B$ iff there is a computable $F : \mathbb{N} \to \mathbb{N}$ with $F^{-1}(B) = A$.

Theorem (Brattka & Pauly)

The many-one degrees embed into $\mathfrak{W}$.

Proof.

Let $p, q \in \mathbb{N}^{\mathbb{N}}$ be Turing incompatible. Map $A \subseteq \mathbb{N}$ to $\chi_A^{p,q} : \mathbb{N} \to \{p, q\}$ where $(\chi_A^{p,q})^{-1}(p) = A$. □

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What really is “and”?

Definition

We call f *join-irreducible*, if $f \leq_W g \sqcup h$ implies that $f \leq_W g$ or $f \leq_W h$.

Most “natural” Weihrauch degrees are join-irreducible.

Definition

Let $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ be defined via $(y, v) \in (f \times g)(x, u)$ iff $y \in f(x)$ and $v \in g(v)$.

Proposition (Brattka)

$(\mathfrak{W}, 0, 1, \sqcup, \times, ^*)$ *is a Kleene-algebra.*

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Let $f \star g = \sup_{\leq_W}\{F \circ G \mid F \leq_W f \wedge G \leq_W g\}$.

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$\star$ actually is a total operation on Weihrauch degrees.

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & Pauly)

$\mathrm{RT}^2_2 \leq_W \mathrm{SRT}^2_2 \star \mathrm{COH}$, *but* RT^2_2 *and* $\mathrm{SRT}^2_2 \times \mathrm{COH}$ *are incomparable.*

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The minimum $\min_{\leq_W}\{h \mid f \leq_W g \star h\}$ *always exists (and is denoted by* $g \to f$*, but in general none of the following have to exist:*

1. $\inf_{\leq_W}\{h \mid f \leq_W h \star g\}$
2. $\inf_{\leq_W}\{h \mid f \leq_W g \times h\}$

This means that the Weihrauch degrees are not a model of any of the usual substructural logics people have studied.

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Closure under composition

Definition (Neumann & Pauly)

An input for $f^\diamond$ is a description of an abstract register machine operating on represented spaces with computable functions and f as operations, together with an input on which the register machine halts. The output is whatever the register machine outputs.

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Characterizations

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f^* is the least Weihrauch degree above f satisfying $1 \leq_W f^*$ and $f^* \times f^* \equiv_W f^*$.

Theorem (Westrick 2020)

$f^\diamond$ is the least Weihrauch degree above f satisfying $1 \leq_W f^\diamond$ and $f^\diamond \star f^\diamond \equiv_W f^\diamond$.

- Open since CCA 2015
- There is a constant function f and a multivalued function g such that $f \leq_W g^\diamond$, but no fixed finite number of applications of g suffices

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$f^\diamond$ is the least Weihrauch degree above f satisfying $1 \leq_W f^\diamond$ and $f^\diamond \star f^\diamond \equiv_W f^\diamond$.

- Open since CCA 2015
- There is a constant function f and a multivalued function g such that $f \leq_W g^\diamond$, but no fixed finite number of applications of g suffices

Algebraic structure, summary

We have the following operations on Weihrauch degrees:

1. $f \sqcap g$, returning either an answer to f or an answer to g (OR)
2. $f \sqcup g$, letting us choose between f and g (AND)
3. $f \times g$, letting us both f and g in parallel (AND)
4. $f \star g$, letting us first use g, then f (AND)
5. $f \to g = \min\{h \mid g \leq_W f \star h\}$ (Implication)
6. $f^*, f^\diamond$ letting us use f finitely many times, in parallel or consecutively (bang, bang)
7. $\widehat{f}$, letting us use f countably many times in parallel (bang)
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Outline

The idea

Sometimes, we can understand a Weihrauch degree by figuring out how it relates to "simple" Weihrauch degrees.

Definition (Dzhafarov, Solomon & Yokoyama)

Let the first-order part of a Weihrauch degree f be:

$$^1 f := \sup_{\leq_W}\{g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N} \mid g \leq_W f\}$$

Definition (Valenti, Goh & Pauly)

Fix a represented space $\mathbf{X}$. The deterministic part of a Weihrauch degree f is

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Some questions and results

Proposition (Hoyrup)

There is an f with $\mathrm{Det}_{\mathbb{N}^\mathbb{N}}(f) <_W \mathrm{Det}_{\mathbb{R}}(f)$.

Proposition (de Brecht, Pauly & Schröder)

For overt choice $\mathbf{VC}_\mathbb{Q} :\subseteq \mathcal{V}(\mathbb{Q}) \rightrightarrows \mathbb{Q}$ *it holds that* ${}^1(\mathbf{VC}_\mathbb{Q}) \equiv_W \mathrm{Det}_{\mathbb{N}^\mathbb{N}}(\mathbf{VC}_\mathbb{Q}) \equiv_W 1$*, but* $\mathbf{VC}_\mathbb{Q}$ *is not computable.*

Question (Valenti, Goh & Pauly)

Is there some f with $\mathrm{Det}_\mathbb{N}(f) <_W \mathrm{Det}_{\mathbb{N}^\mathbb{N}}({}^1 f)$*? (It always holds that* $\mathrm{Det}_\mathbb{N}(f) \equiv_W {}^1\,\mathrm{Det}_{\mathbb{N}^\mathbb{N}}(f)$*)*

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Outline

The Weihrauch lattice

Structures embeddable in the Weihrauch degrees

More algebraic operations

Special subclasses

Some side comments

The big open questions

Irreducibility

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There are $f, g <_W \lim$ *with* $f \times g \equiv_W \lim$*.*

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- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into the Weihrauch degrees (via the Medvedev degrees).
- Thus, any universally quantified statement using $\sqcup$ and $\sqcap$ is either provable from the axioms of distributive lattices or false in $\mathfrak{W}$.
- Can we extend this to additional operations?
- A list of known axioms and non-axioms is available in "On the algebraic structure of Weihrauch degrees", LMCS 2018

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Continuous Weihrauch reducibility

If we relativize Weihrauch reducibility relative to an arbitrary oracle, we get continuous Weihrauch reducibility.

Question

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