Synthetic mathematics with an excursion into computability theory

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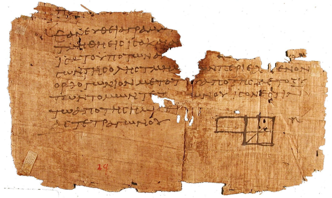
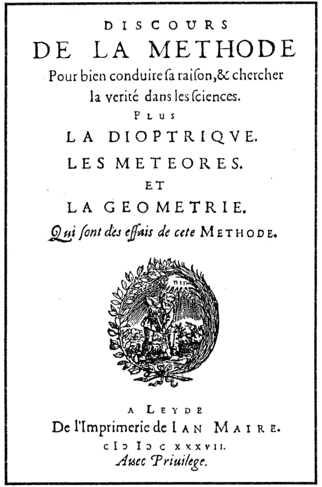
1. Thank you for the invitation. I am delighted to have the opportunity to speak at your seminar. Hopefully one day I can also visit.





DISCOURS
DE LA METHODE
Pour bien conduire sa raison, & chercher
la verité dans les sciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui sont des essais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
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Auec Priuilege.

1. Today I would like to speak about *synthetic mathematics*, and specifically about *synthetic computability*.
2. Let us begin by explaining what is meant by "synthetic". It is often said that Euclid's geometry is *synthetic* whereas Descartes's geometry is *analytic*.

Synthetic:

- Basic objects are taken as *primitive*.
- Their properties & relations are *axiomatized*.
- We work within the axiomatic system.

Analytic:

- Basic objects are *constructed* from other objects.
- Their properties & relations are *deduced*.
- We work in a wider mathematical environment.

1. The synthetic approach takes the basic objects as primitive, and *axiomatizes* their properties and relations. All reasoning relies just on the axioms.
2. Example: Euclid's geometry takes basic notions (point and lines) as primitive and relies on the axiomatic method. (NB: lines are *not* sets of points!)
3. In the analytic approach the basic objects are built from other objects. Their properties and relations are *deduced*.
4. Example: Descartes's geometry employs a coordinate system, represents points as elements of $\mathbb{R}^2$, and lines as certain subsets of $\mathbb{R}^2$.
5. Each approach has its advantages. The synthetic one often exposes the essence of an argument, while the analytic one provides a wider set of methods by which we may attack a problem.

Synthetic differential geometry:

"All maps are smooth!"

Synthetic topology:

"All maps are continuous!"

Synthetic computability:

"All maps are computable!"

1. Euclid's geometry is about a single structure – the Euclidean plane. We seek to develop the synthetic method for an *entire branch* of mathematics, such as differential geometry, or topology, or computability theory.
2. In fact these have all been developed, as well as others. We cannot give a proper historical account here, but suffice let me mention some names.
3. Synthetic differential geometry is probably the oldest, and has been developed by Eduardo Dobuc, Bill Lawvere, and Anders Kock. In it all maps are smooth and nilpotent infinitesimals exist. Robinson's infinitesimal analysis is similar, but does not have nilpotent infinitesimals. It also can be viewed as a kind of synthetic mathematics.
4. Synthetic topology has been studied by Martin Escardó, Paul Taylor, Davorin Lešnik, and myself. It builds on the work done in the 1980's on synthetic domain theory. The idea here is that "every set has an intrinsic topology" and "all maps are continuous".
5. I shall speak about synthetic computability in detail.

Synthetic

Analytic

1. As logicians, we may formulate the situation in familiar terms. “Synthetic” is “theory”, “analytic” the “model”, and they are related by an interpretation. We shall adopt this view.
2. There are several options for what “theory” and “model” mean here.
3. For the geometry of the plane we could take a first-order theory and a first-order structure.
4. To capture an entire branch of mathematics, we need a sufficiently complex notion of model, as well as an expressive language. Higher-order logic (often intuitionistic) and toposes fit the bill.
5. If you are not familiar with these, do not despair. The differences between set theory and toposes are largely superficial. Also informal higher-order logic and set theory are quite similar, and I shall explicitly describe the model. If anyone is interested in the technical details, we can discuss them too.

Synthetic Analytic

$$\text{Theory} \xrightarrow{\text{interpretation}} \text{Model}$$

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$$\text{Theory} \xrightarrow{\text{interpretation}} \text{Model}$$

$$\text{HOL} \xrightarrow{\text{internal language}} \text{Topos}$$

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$$\text{IHOL} \xrightarrow{\text{realizability}} \text{Effective topos}$$

The method:

1. Take a classic theorem in computability theory.
2. Rephrase it as a fact about the effective topos.
3. Find a statement whose interpretation is the fact.
4. Abstract the statement to expose its essence.
5. Give a synthetic proof.

1. Our model of choice is the effective topos, whose internal language is intuitionistic higher-order logic, and the interpretation is Kleene's realizability.
2. We shall take a closer look at these ingredients, but let us first explain the general method by which synthetic results are obtained.
3. The first three steps are best done in the privacy of one's notebook, as they are mostly just technique. The last two steps are most interesting, because they show how synthetic mathematics works.

The effective topos

1. The effective topos was discovered by Martin Hyland in the 1980’s.
2. It is a bit complicated to describe. Luckily, there is a subcategory of it, which is quite intuitive, easier to describe, and sufficient for our purposes. So let us do that instead.

Assemblies

Definition

An *assembly* $A = (|A|, \Vdash_A)$ is a set $|A|$ with a *realizability* relation $\Vdash_A \subseteq \mathbb{N} \times |A|$ such that $\forall x \in A \,.\, \exists n \in \mathbb{N} \,.\, n \Vdash_A x$.

When $n \Vdash_A x$ we say that n *realizes* x. Think of n as the Gödel code of x.

Definition

An *assembly map* $f : A \to B$ is a function $f : |A| \to |B|$ for which there exists $k \in \mathbb{N}$ such that

$$\forall x \in |A| \,.\, \forall n \in \mathbb{N} \,.\, n \Vdash_A x \Rightarrow \varphi_k(n) \Vdash_B f(x).$$

Above φ_k is the k-th partial computable map. We say that φ_k *tracks* or *realizes* f.

1. An assembly is quite literally the formal expression of the idea that mathematical objects may be coded with numbers.
2. Note that every element must have at least one realizers, and may have several.
3. We also allow the same number to code several elements. While this sounds unusual, it is quite useful.
4. The appropriate notion of morphism between assemblies is that of a map that is tracked by a partial computable map.

Examples of assemblies

1. *Natural numbers:* $\mathsf{N} = (\mathbb{N}, \Vdash_{\mathsf{N}})$
   where $k \Vdash_{\mathsf{N}} n \iff k = n$.
2. *Computable maps:* $\mathsf{N}^{\mathsf{N}} = (\mathcal{R}, \Vdash_{\mathsf{N}})$
   where $\mathcal{R}$ is the set of computable maps $\mathbb{N} \to \mathbb{N}$ and
   $n \Vdash_{\mathbb{N}^{\mathbb{N}}} f \iff f = \varphi_n$.
3. *Computably enumerable sets:* $\mathsf{E} = (\mathsf{CE}, \Vdash_{\mathsf{E}})$
   where CE is the set of c.e. sets and
   $n \Vdash_{\mathsf{E}} S \iff S = \{k \in \mathbb{N} \mid \varphi_n(k)\downarrow\}$.

1. The well-known objects in computability theory form assemblies.
2. Any set X may be equipped with the trivial computability structure in which every number realizes every element. We get a full and faithful functor $\nabla : \mathsf{Set} \to \mathsf{Asm}$ of sets into assemblies. The assembly ∇X is used to *non-uniformly* parameterize with X a theorem or a construction.

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4. *Classical sets:* $\nabla X = (X, \Vdash_{\nabla X})$
   where X is *any* set and
   $n \Vdash_{\nabla X} x$ for all $n \in \mathbb{N}$ and $x \in X$.

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Two-element assemblies

1. *The boolean truth values:* $\mathsf{2} = (\{\bot, \top\}, \Vdash_{\mathsf{2}})$ where $0 \Vdash_{\mathsf{2}} \bot$ and $1 \Vdash_{\mathsf{2}} \top$.
2. *The semidecidable truth values:* $\mathbb{S} = (\{\bot, \top\}, \Vdash_{\mathbb{S}})$ where $n \Vdash_{\mathbb{S}} \bot \iff n \notin K$ and $n \Vdash_{\mathsf{2}} \top \iff n \in K$. ($K = \{n \in \mathbb{N} \mid \varphi_n(n){\downarrow}\}$ is the Halting set.)
3. *The classical truth values:* $\nabla\mathsf{2} = (\{\bot, \top\}, \Vdash_{\nabla\mathsf{2}})$ where $n \Vdash_{\nabla\mathsf{2}} b$ for all $n \in \mathbb{N}$ and $b \in \{\bot, \top\}$.

1. Just like there are several topologies on a two-element set, there are many assembly structures on it. In fact, many more than there are topologies.
2. The two-element assemblies encode various notions of decision procedures.

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Decision procedures/subsets:

1. Decision procedure/subset: $A \to \mathsf{2}$.
2. Semi-decision procedure/subset: $A \to \mathbb{S}$.
3. Non-computational decision/subset: $A \to \nabla\mathsf{2}$.

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Realizability interpretation of logic

$n \Vdash \top$	always
$n \Vdash \bot$	never
$n \Vdash a =_A b$	if $a, b \in \lvert A \rvert$ and $a = b$ and $n \Vdash_A a$,
$\langle m, n \rangle \Vdash \phi \wedge \psi$	if $m \Vdash \phi$ and $n \Vdash \psi$,
$\langle 0, n \rangle \Vdash \phi \vee \psi$	if $m \Vdash \phi$,
$\langle 1, n \rangle \Vdash \phi \vee \psi$	if $m \Vdash \psi$,
$n \Vdash \phi \Rightarrow \psi$	if $k \Vdash \phi$ implies $\varphi_n(k) \Vdash \psi$,
$n \Vdash \forall x \in A \,.\, \phi(x)$	if $a \in \lvert A \rvert$ and $k \Vdash_A a$ implies $\varphi_n(k) \Vdash \phi(a)$,
$\langle m, n \rangle \Vdash \exists x \in A \,.\, \phi(x)$	if there is $a \in \lvert A \rvert$ such that $m \Vdash_A a$ and $n \Vdash \phi(a)$.

1. The interpretation of intuitionistic logic in the effective topos is Kleene's *realizability* relation, sketched out here.
2. It is essentially the computability-theoretic expression of the Brouwer-Heyting-Kolmogorov reading of the meaning of constructive logic. Rather than explaining when a statement is true, we explain which numbers realize it. The truth value of a statement is then the set of numbers that realize it.
3. If a statement is provable in intuitionistic logic then it has a realizer (but not vice versa).
4. The realizability interpretation converts proofs carried out in intuitionistic logic to computations.
5. For example, a proof of $\forall x \in A \,.\, \phi(x) \vee \psi(x)$ translates to a *decision procedure* which, given $n \Vdash_A x$ outputs $\langle 0, k \rangle$ or $\langle 1, m \rangle$ from which me may discern which of the two disjuncts holds, and why.
6. The upshot is that in the synthetic world we never have to explicitly mention any computations, because the realizability interpretation reconstructs them from proofs.

Two-element sets – synthetically

- Every topos has the object of truth values Ω.
- The power set of A is $\mathcal{P}A = \Omega^A$.
- Decidable truth values:

$$\mathsf{2} = \{p \in \Omega \mid p \vee \neg p\}.$$

- The semi-decidable truth values:

$$\mathbb{S} = \{p \in \Omega \mid \exists f : \mathsf{2}^{\mathbb{N}} \,.\, (p \iff \exists n \in \mathbb{N} \,.\, f(n))\}.$$

- The classical truth values:

$$\nabla\mathsf{2} = \{p \in \Omega \mid \neg\neg p \Rightarrow p\}.$$

1. Here is our first exercise: construct the various two-element assemblies considered earlier synthetically. We shall refer to the objects of the effective topos as *sets* to keep things simple.
2. Every topos has a subobject classifier, or the set of truth values. In the effective topos, it is *not* an assembly, we shall just take it for granted.

Theorem (Lawvere)

If $e : A \twoheadrightarrow B^A$ is surjective then B has the fixed point property: for every $f : B \to B$ there is $x_0 \in B$ such that $f(x_0) = x_0$.

Proof.

Given $f : B \to B$, define $g(y) = f(e(y)(y))$. Because e is surjective there is $x \in A$ such that $e(x) = g$. Then $e(x)(x) = f(e(x)(x))$, so $x_0 = e(x)(x)$ is a fixed point of f. □

1. The realizability interpretation converts constructive proofs to computations.
2. Thus, some synthetic computability theorems are "free" in the sense that they require no special axioms.
3. The realizability translation of the corollary is: there is no computable enumeration of computable total maps.

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Corollary

There is no surjection $e : \mathbb{N} \twoheadrightarrow \mathbb{N}^{\mathbb{N}}$.

Proof.

The successor map does not have a fixed point. □

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Proposition (Rice's theorem)

If A has the fixed point property then every map $A \to \mathsf{2}$ is constant.

Proof.

Given $f : A \to \mathsf{2}$ and any $x, y \in A$ we show that $f(x) = f(y)$. Define $g : A \to A$ by

$$g(z) = \begin{cases} x & \text{if } f(z) = f(y), \\ y & \text{otherwise.} \end{cases}$$

There is $u \in A$ such that $u = g(u)$. If $f(u) = f(y)$ then $u = g(u) = x$ hence $f(x) = f(u) = f(y)$. If $f(u) \neq f(y)$ then $u = g(u) = y$ and so $f(u) = f(y)$, a contradiction, hence again $f(x) = f(y)$. □

Here is another "free theorem", namely Rice's theorem.This is perhaps a bit difficult to relate to the traditional Rice's theorem. We shall do so later. For now, observe that we have extracted the *essence* of Rice's theorem: the fixed-point property.

Enumerable & finite sets

- A is *finite* if there exist $n \in \mathbb{N}$ and a surjection

$$e : \{1, \ldots, n\} \twoheadrightarrow A,$$

called a *listing* of A. An element may be listed more than once.

1. Recall the following basic notions.

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- A is *infinite* if there exists an injective $i : \mathbb{N} \rightarrowtail A$.

1. Recall the following basic notions.

Multi-valued maps

Definition

A *multi-valued map* $f : A \rightrightarrows B$ is a map $f : A \to \mathcal{P}B$ such that $\forall x \in A \,.\, \exists y \in B \,.\, y \in f(x)$.

The multi-valued maps $f : A \rightrightarrows B$ are in correspondence with total relations $R \subseteq A \times B$.

1. It is convenient to work with multi-valued maps, i.e., those that can return many results.

Axiom of Choice

- Axiom of Choice is *not* valid in the effective topos:

  Every $f : A \rightrightarrows B$ *has a choice function* $g : A \to B$ *such that* $g(x) \in f(x)$ *for all* $x \in A$.

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- *Number Choice* is valid:

  Every $f : \mathbb{N} \rightrightarrows B$ has a choice function $g : \mathbb{N} \to B$.

- *Dependent Choice* is valid:

  Given $x \in A$ and $h : A \rightrightarrows A$, there exists $g : \mathbb{N} \to A$ such that $g(0) = x$ and $g(n+1) \in h(g(n))$ for all $n \in \mathbb{N}$.

  This is a form of *simple recursion* for multi-valued functions.

1. The general axiom of choice is invalid in the effective topos, but number choice and dependent choice are realized.

Theorem (Recursion Theorem)

If there is a surjection $e : \mathbb{N} \to A^{\mathbb{N}}$ then every multi-valued map $f : A \rightrightarrows A$ has a fixed point, which is an $x \in A$ such that $x \in f(x)$.

Proof.

For every $n \in \mathbb{N}$ there is $x \in f(e(n)(n))$, hence by Countable Choice there is a map $g : \mathbb{N} \to A$ such that $g(n) \in f(e(n)(n))$ for all $n \in \mathbb{N}$. There is $k \in \mathbb{N}$ such that $e(k) = g$, and so $e(k)(k) = g(k) \in f(e(k)(k))$. □

1. The recursion theorem is a fixed-point principle for multi-valued maps.
2. Note that we still need on special axioms to prove the theorem, apart from number choice.
3. We will need extra axioms to find interesting instances of the theorem. Classically there are none.

Axiom of Enumerability

For $S \in \mathcal{P}\mathbb{N}$, let $\mathsf{enumerable}(S)$ be the the predicate

$$\exists f \in \mathbb{N}^{\mathbb{N}} \,.\, \forall n \in \mathbb{N} \,.\, (n \in S \iff \exists k \in \mathbb{N} \,.\, f(k) = n + 1)$$

Define the set of all enumerable subsets of $\mathbb{N}$:

$$\mathcal{E} = \{S \in \mathcal{P}\mathbb{N} \mid \mathsf{enumerable}(S)\}.$$

Axiom (Enumerability)

There are enumerably many enumerable sets of numbers.

Let $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E}$ be an enumeration.

1. The central axiom of synthetic computability sounds crazy from a classical viewpoint.
2. However, it is valid in the effective topos. It states that there exists a computable enumeration of c.e. sets.

First consequences

Proposition

The enumerable and semi-decidable subsets of $\mathbb{N}$ coincide: $\mathcal{E} \cong \mathbb{S}^{\mathbb{N}}$.

(We leave the proof as exercise. It uses Number Choice.)

Proposition

$\mathbb{S}$ and $\mathcal{E}$ have the fixed-point property.

Proof.

By Lawvere, $\mathsf{W} : \mathbb{N} \twoheadrightarrow \mathcal{E} = \mathbb{S}^{\mathbb{N}} \cong \mathbb{S}^{\mathbb{N}\times\mathbb{N}} \cong \mathcal{E}^{\mathbb{N}}$. □

The Law of Excluded Middle Fails

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Corollary

The Law of Excluded Middle is false.

Proof.

Among the sets $\mathsf{2} \subseteq \mathbb{S} \subseteq \Omega$ only the middle one has the fixed-point property, so $\mathsf{2} \neq \mathbb{S} \neq \Omega$. □

Post's theorem

Axiom (Markov's principle)

- *Traditional:*
  If a binary sequence is not constantly 0 *then it contains a* 1.
- *Synthetic:* $\mathbb{S} \subseteq \nabla \mathsf{2}$.

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- *Synthetic:* $\mathbb{S} \subseteq \nabla 2$.

Theorem (Post)

- *Traditional:*
  $A \subseteq \mathbb{N}$ *is decidable if* A *and* $\mathbb{N} \setminus A$ *are semidecidable.*
- *Synthetic:* $2 = \{p \in \Omega \mid p \in \mathbb{S} \wedge \neg p \in \mathbb{S}\}$.

Many-to-one reducibility

Definition

A *many-to-one* reduction from $S \subseteq \mathbb{N}$ to $T \subseteq \mathbb{N}$ is a map $r : \mathbb{N} \to \mathbb{N}$ such that $S = \{n \in \mathbb{N} \mid r(n) \in T\} = r^{-1}(T)$.

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Proposition

$K = \{\langle m, n \rangle \mid m \in \mathsf{W}_n\}$ *is many-to-one complete for* $\mathcal{E}$.

Proof.

If $S \in \mathcal{E}$ then $S = \mathsf{W}_n$ for some n, hence for all $m \in \mathbb{N}$,
$m \in S \iff m \in \mathsf{W}_n \iff \langle m, n \rangle \in K$. □

Dcpo's and ω-cpos

A poset $(P, \leq)$ is:

- *chain-complete (ω-cpo)* if every chain (increasing sequence) in P has a supremum.
- *directed-complete (dcpo)* if every directed subset of P has a supremum.

A *base* for $(P, \leq)$ is an enumerable subset $B \subseteq P$ with decidable equality, such that:

1. in ω-cpos: every element is the supremum of a chain in B.
2. in dcpos: every element is the supremum of a directed subset of B.

Partial oracles: $\mathbb{O} = \{(S_0, S_1) \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \mid S_0 \cap S_1 = \emptyset\}$.
Plotkin's domain: $\mathbb{T}^\omega = \{(S_0, S_1) \in \mathcal{E} \times \mathcal{E} \mid S_1 \cap S_1 = \emptyset\}$.
Both are ordered by component-wise $\subseteq$.

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Observations:

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Theorem

Every $f : \mathbb{T}^\omega \to \mathbb{T}^\omega$ is preserves suprema of chains.

Turing reducibility

Definition

A *Turing reduction* is map $r : \mathbb{O} \to \mathbb{O}$ which factors through $\mathbb{T}^\omega$:

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Say that $S \in \max \mathbb{O}$ is is *Turing-reducible* to $T \in \max \mathbb{O}$ if there is a reduction $r : \mathbb{O} \to \mathbb{O}$ such that $S = r(T)$.

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Exercise: give a synthetic proof of Friedberg-Mučnik theorem.