Coideals and the Local Ramsey Property - Preliminary -

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Ramsey's theorem 1929

Notation: $A^{[n]} = \{B \subseteq A : |B| = n\}$, $A^{[<\infty]} = \bigcup_n A^{[n]}$

$$A^{[\infty]} = \{B \subseteq A : |B| = \infty\}$$

Infinite version: For every finite coloring of $\mathbb{N}^{[2]}$ and every $A \in \mathbb{N}^{[\infty]}$ there is $B \in A^{[\infty]}$ such that $B^{[2]}$ is monochromatic.

Ramsey's theorem 1929

Notation: $A^{[n]} = \{B \subseteq A : |B| = n\}$, $A^{[<\infty]} = \bigcup_n A^{[n]}$

$$A^{[\infty]} = \{B \subseteq A : |B| = \infty\}$$

Generalized infinite version: Given an integer $n > 0$, for every finite coloring of $\mathbb{N}^{[n]}$ and every $A \in \mathbb{N}^{[\infty]}$ there is $B \in A^{[\infty]}$ such that $B^{[n]}$ is monochromatic.

Ramsey property

Question: Given $X \subseteq \mathbb{N}^{[\infty]}$, is there $A \in \mathbb{N}^{[\infty]}$ such that $A^{[\infty]} \subseteq X$ or $A^{[\infty]} \cap X = \emptyset$?

Answer: Not in general.

Example: For $A, B \in \mathbb{N}^{[\infty]}$, $\quad A \sim B$ iff $|A \triangle B| < \infty$

(AC) Pick an element B_x of each class $x \in \mathbb{N}^{[\infty]}/\sim$,

Let $cl(A)$ denote the class of A and define

$$X = \{A \in \mathbb{N}^{[\infty]} : |A \triangle B_{cl(A)}| \text{ is even}\}$$

Metric Topology on $\mathbb{N}^{[\infty]}$

Identify each $A \in \mathbb{N}^{[\infty]}$ with the increasing sequence $\{A(j)\}_j$ of its elements. Define the metric d on $\mathbb{N}^{[\infty]}$ by:

$$d(A,B) = \begin{cases} 0 & \text{if} \quad A = B \\ \frac{1}{n+1} & \text{if} \quad n = min\{j : A(j) \neq B(j)\} \end{cases}$$

Basic open sets: $[a] = \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B\}$, where $a \in \mathbb{N}^{[<\infty]}$.

The Ramsey Property

A set $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is said to be **Ramsey** if for every $A \in \mathbb{N}^{[\infty]}$ there exists $B \in A^{[\infty]}$ such that $B^{[\infty]} \subseteq \mathcal{X}$ or $B^{[\infty]} \subseteq \mathcal{X}^c$.

EXAMPLES:
(1) The set $\mathcal{X} = \{B \in \mathbb{N}^{[\infty]} : \mathit{min}\ B = 8\}$ *is* Ramsey.
(2) Let $a \subset \mathbb{N}$ be a finite subset. The basic metric set $\mathcal{X} = [a] = \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B\}$ is Ramsey.
NOTE: These are open sets. Notice that $\mathcal{X}^c$ (which is closed) is also Ramsey.

The Ramsey Property

- Clopen sets are Ramsey (Nash-Williams, 1965). Recall that a set is clopen if it is both closed and open.
- Open sets are Ramsey (Galvin,1968)
- Borel sets are Ramsey (Galvin and Prikry, 1973). A set is said to be Borel if it is an element of the σ-algebra generated by the collection of all open sets.

NOTE: Silver (1972) proved that *analytic* sets (continuous images of Borel sets) are Ramsey, but using *metamathematical* techniques.

The Ramsey Property

Ellentuck's topology on $\mathbb{N}^{[\infty]}$:

$$[a, A] = \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\},$$

where $A \in \mathbb{N}^{[\infty]}$ and $a \subset A$ is finite.

A set $X \subseteq \mathbb{N}^{[\infty]}$ is said to be **completely Ramsey** if for every nonempty $[a, A]$ there is $B \in [a, A]$ such that $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$. X is said to be **completely Ramsey null** if for every nonempty $[a, A]$ there is $B \in [a, A]$ such that $[a, B] \cap X = \emptyset$.

The Ramsey Property

Theorem: (Ellentuck, 1974) Let $X \subseteq \mathbb{N}^{[\infty]}$ be given. Then,

1. X is completely Ramsey if and only if X has the Baire property in Ellentuck's topology.
2. X is completely Ramsey null if and only if X is meager in Ellentuck's topology.

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1. Local Ramsey Property

Local Ramsey Property

- A family H $\subseteq \mathbb{N}^{[\infty]}$ is a **coideal** if it satisfies the following:

(i) A $\subseteq$ B and A $\in$ H implies B $\in$ H;

(ii) A$\cup$B $\in$ H implies A $\in$ H or B $\in$ H.

- $[a$, A] = {B $\in \mathbb{N}^{[\infty]}$: B $\subseteq$ A and $a \sqsubset B$}

- (Mathias) Let H $\subseteq \mathbb{N}^{[\infty]}$ be a coideal. X $\subseteq \mathbb{N}^{[\infty]}$ is H-**Ramsey** if for every non empty [a,A] with A $\in$ H there exists B $\in$ [a,A]$\cap$H such that [a,B] $\subseteq$ X or [a,B]$\cap$X = $\emptyset$. X is H-**Ramsey null** if for every non empty [a,A] with A $\in$ H there exists B $\in$ [a, A] $\cap$ H such that [a, B] $\cap$ X = $\emptyset$.

Local Ramsey Property

- A coideal H is **selective** if and only if every decreasing sequence

$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ in H has a *diagonalization* in H: i.e., there is $B \in H$ such that $B/n \subseteq A_n$ for all $n \in B$.

- H is **semiselective** if for every sequence $\{D_n\}_n$ of dense open subsets

in $(H, \subseteq^*)$, the family of its diagonalizations is dense: i.e., for every

$A \in H$ there is $B \in H$ such that $B \subseteq^* A$ and $B/n \in D_n$ for all $n \in B$.

Local Ramsey Property

- (Ellentuck, 1974). Let $X \subseteq \mathbb{N}^{[\infty]}$ be given.

(i) X is Ramsey if and only if X has the Baire property.

(ii) X is Ramsey null if and only if X is meager.

- (Mathias, 1977). Let $X \subseteq \mathbb{N}^{[\infty]}$ and let H be a selective coideal.

(i) X is H-Ramsey if and only if X has the H-Baire property.

(ii) X is H-Ramsey null if and only if X is H-meager.

- (Farah, 1997). Let H be a coideal. The following are equivalent:

(i) H is semiselective.

(ii) The H-Ramsey subsets of $\mathbb{N}^{[\infty]}$ are exactly those sets having the H- Baire property …

2. Local Ramsey property in terms of games

***Ideal games and Ramsey sets*,**
jointly with Carlos Di Prisco and Carlos Uzcátegui.
(2012. Proc. of the Amer. Math. Soc.)

Infinite game $G_H(a, A, X)$ *(Kastanas, Matet)*

Fix a coideal $H \subseteq \mathbb{N}^{[\infty]}$, $X \subseteq \mathbb{N}^{[\infty]}$, $A \in H$ and $a \in \mathbb{N}^{[<\infty]}$

I: A_0 A_1 $\cdots$ $A_k \cdots$

II: (n_0, B_0) (n_1, B_1) $\cdots$ (n_k, B_k) $\cdots$

$A_0 \in H \cap [a, A]$; $A_k, B_k \in H$; $n_k \in A_k$, $B_k \subseteq A_k / n_k$; $A_{k+1} \subseteq B_k$

Player I wins if and only if $a \cup \{n_0, n_1, n_2 \ldots\} \in X$

Local Ramsey property in terms of games

- (Kastanas, 1983). X is Ramsey if and only if for every A $\in \mathbb{N}^{[\infty]}$ and

$a \in \mathbb{N}^{[<\infty]}$ the game $G_{\mathbb{N}[\infty]}(a, A, X)$ is determined.

- (Matet, 1993). Let H be a selective coideal. X is H-Ramsey if and only if for every A $\in$ H and $a \in \mathbb{N}^{[<\infty]}$ the game $G_H(a, A, X)$ is determined.

Semiselectivity?

- (DP – M – U) Let H be a coideal. The following are equivalent:

(i) H is semiselective.

(ii) For every $X \subseteq \mathbb{N}^{[\infty]}$, X is H-Ramsey if and only if for every $A \in H$ and $a \in \mathbb{N}^{[<\infty]}$ the game $G_H(a, A, X)$ is determined.

REMARK

- The phenomena described so far have analogs in other contexts: structures where *a Ramsey property* can be defined and characterized in *Ellentuck-like* terms are known as *Topological Ramsey Spaces.*

(***Will define formally later***).

- We gave an abstract approach to the local Ramsey property within the framework of topological Ramsey space.

(Will introduce this later).

3. Local Ramsey "Theories" of Block Sequences

Ramsey sets of block sequences of vectors, jointly with Daniel Calderon and Carlos Di Prisco
(2021. Submitted)

Block sequences

- $\mathrm{FIN}^{[\infty]}$ = block sequences of non empty finite sets
- $\mathrm{FIN}_k^{[\infty]}$ = block sequences of "vectors"

$p:\mathbb{N}\to\{0,1,...,k\}$ such that $\mathrm{supp}(p) = \{n : p(n) \neq 0\}$ is finite and $k \in \mathrm{range}(p)$

$\mathrm{FIN}^{[\infty]}$ and $\mathrm{FIN}_k^{[\infty]}$ are topological Ramsey spaces

(by Milliken and Todorcevic, respectively)

So our abstract local Ramsey theory applies. Yet…

Comparing local Ramsey theories for block sequences

- (Blass) An ultrafilter U on FIN is an **ordered-union ultrafilter** if it has a basis of sets of the form FU(A) where $A \in FIN^{[\infty]}$.

U is said to be **stable** if for every sequence $\{D_n\}_n \subseteq FIN^{[\infty]}$ such that $FU(D_n) \in U$ for every n, there is $E \in FIN^{[\infty]}$ such that $FU(E) \in U$ and for every n $E \leq^* D_n$.

Comparing local Ramsey theories for block sequences

- (Eisworth) A family $H \subseteq FIN^{[\infty]}$ is **Matet-adequate** if

(1) H is closed under finite changes,

(2) For all A, B $\in FIN^{[\infty]}$, if A $\in$ H and A $\leq$ B then B $\in$ H.

(3) (H, $\leq^*$) is σ-closed,

(4) If A $\in$ H and FU(A) is partitioned into 2 pieces then there is B $\leq$ A in H so that FU(B) is included in a single piece of the partition (this is called *the Hindman property*).

Comparing local Ramsey theories for block sequences

- *Stable ordered-union ultrafilter* on FIN vs *selective ultrafilter* on $\mathrm{FIN}^{[\infty]}$
- *Mate adequate family* on $\mathrm{FIN}^{[\infty]}$ vs *selective coideal* on $\mathrm{FIN}^{[\infty]}$

Comparing local Ramsey theories for block sequences

- (Blass) For any ordered union ultrafilter U on FIN, the following are equivalent.

(1) U is stable.

(2) $U^{\infty} = \{A \in FIN^{[\infty]} : FU(A) \in U\}$ is selective

(3) U has the Ramsey property for pairs

Comparing local Ramsey theories for block sequences

(C – DP – M)

- The generalization of Blass' result to FIN_k holds.
- Matet-adequate families and selective coideals on $\mathrm{FIN}^{[\infty]}$ coincide. The corresponding generalizations to $\mathrm{FIN}_k^{[\infty]}$ also coincide.

4. Abstract Approach to the Local Ramsey Property

- ***A notion of selective ultrafilter corresponding to Topological Ramsey Spaces*.** (2007. Math. Logic Quarterly)
- ***Local Ramsey theory, and abstract approach,*** jointly with Carlos Di Prisco and Jesús Nieto.
  (2017. Math. Logic Quarterly)

Topological Ramsey spaces

- (R, ≤, r)

r : $\mathbb{N}$ × R → AR;   r(n, A) is the n-th approximation of A

For a in AR and A in R, $[a$, A] = {B in R : B ≤ A and r(n, B) = a};

(Use to define ***Ramsey set*** like in $\mathbb{N}^{[\infty]}$)

Todorcevic introduces axioms **A1, A2, A3** and **A4** for a structure (R, ≤, r).

A1, A2 permit to understand R as a metric subspace of the Polish space $AR^{\mathbb{N}}$. They also make the family of sets $[a$, A] a base for another topology on R (Ellentuck-like).

A3 makes R a closed subset of $AR^{\mathbb{N}}$.

A4 says that (R, ≤, r) satisfies a "pigeon hole principle".

Topological Ramsey spaces

- $(R, \leq, r)$ is a **topological Ramsey space** if Baire sets and Ramsey sets coincide (i.e., "Ellentuck's theorem" holds).

(Todorcevic) If $(R, \leq, r)$ satisfies **A1** – **A4**, then it is a Topological Ramsey space.

Abstract coideals

- Given (R, ≤, r) satisfying **A1** − **A4**, a subset H ⊆ R is a **coideal** if:

(a) A∈H and A≤B implies B∈H.

(b) H satisfies a local version of **A3**.

(c) H satisfies a local version of the pigeon hole principle **A4**.

Abstract coideals

- **Almost reduction**:

For A, B ∈ R, write A $\leq^*$ B if there exists an approximation a such that

$\emptyset \neq [a, \mathrm{A}] \subseteq [a, \mathrm{B}]$

With these definitions, the notions of *H*-***Ramsey*** set, *H*-***Baire*** set, ***dense open*** *in (H, $\leq^*$),* ***selective coideal*** and ***semiselective coideal*** can be lifted to the framework of the topological Ramsey space (R, ≤, r) .

Local Ramsey property, captured abstractly.

(DP – M – N) Given (R, ≤, r) satisfying **A1** − **A4** , if H ⊆ R *is a coideal, then the following statements are equivalent:*

(1) H *is a semiselective.*

(2) X ⊆ R *is* H–*Ramsey iff* X *is* H–*Baire.*

(3) X ⊆ R *is* H–*Ramsey null iff* X *is* H–*Meager.*

REMARK

ultrafilter = *maximal filter on* (R, ≤) satisfying local versions of **A3** and **A4**.

If we don't assume local versions of **A3** and **A4**, then…

- (Trujillo) It is possible to show the existence of an ultrafilter that is selective but not Ramsey!

(We don't want that!) We want: *Selective* → *Semiselective* → *Ramsey*
… And we get it if we add **A3** and **A4**.

Interesting consequences

(DP – M – N)

- Forcing with (H, $\leq^*$) adds no new elements R, and if U is the
  (H,$\leq^*$)–generic filter over some ground model V, then U is a selective
  ultrafilter in V[U].
- If there exists a super compact cardinal, then every selective ultrafilter U $\subseteq$ R is (R,$\leq^*$)– generic over L($\mathbb{R}$).
- If there exists a super compact cardinal and H $\subseteq$ R is a semiselective coideal, then all definable subsets of R are H–Ramsey.

Next?

- Abstract infinite games. Play the game in your favorite topological Ramsey space.
- Does this work in “non sequential” Ramsey spaces? Reference:

***Topological Ramsey Spaces and Metrically Baire Sets*.**

Jointly with Natasha Dobrinen.

2015. Journal of Combinatorial Theory Series A.

Thank you all!