Big Ramsey degrees for finite binary classes with finitely many forbidden structures

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Recall Ramsey's theorem:

Theorem (Ramsey 1930)

Let $n, r < \omega$. Then

$$\aleph_0 \rightarrow (\aleph_0)^n_r$$

meaning that for any coloring γ of $[\aleph_0]^n$ into r colors, there is an infinite $X \subseteq \omega$ with $|\gamma[[X]^n]| = 1$.

How to generalize? Can change the cardinals which appear.

Theorem (Erdős-Rado 1956)

Let $n < \omega$, and let κ be an infinite cardinal. Then

$$(\beth_{n-1}(\kappa))^+ \rightarrow (\kappa^+)^n_\kappa$$

Another generalization: color finite substructures of a given infinite structure, while demanding that the "monochromatic set" is a structure specified in advance.

Example: the Rado graph

Suppose we color the vertices of the Rado graph in finitely many colors. Can we find an infinite induced subgraph isomorphic to the Rado graph whose points all receive one color? YES (not difficult)

Suppose instead we color the edges. Then the answer is NO.

Erdős, Hajnal, and Pósa (1973) give the following 2-coloring of the edges of the Rado graph.

Enumerate the vertices, say $R = \{v_n : n < \omega\}$. Define $s_n \in 2^{<\omega}$ by declaring $s_n \in 2^n$, and for $m < n$, we have $s_n(m) = 1$ iff $E(v_m, v_n)$.

Now define $\gamma\colon E(R) \to 2$ as follows. Suppose $E(v_m, v_n)$ with $m < n$. We set $\gamma(v_m, v_n) = 0$ iff $s_m \preceq_{lex} s_n$, and $\gamma(v_m, v_n) = 1$ otherwise.

Then any induced subgraph isomorphic to the Rado graph must contain both colors of edges.

Remarkably, 2 colors is the worst possible.

Theorem (Pouzet, Sauer (1996))

For any $\ell < \omega$ and any coloring $\gamma\colon E(R) \to \ell$, there is $X \subseteq R$ which induces a copy of the Rado graph all of whose edges receive at most 2 of the colors.

About a decade later, Sauer proved an analogous result for every finite graph. If A is a finite graph, let $\binom{R}{A}$ denote the copies of A inside R.

Theorem (Sauer (2006))

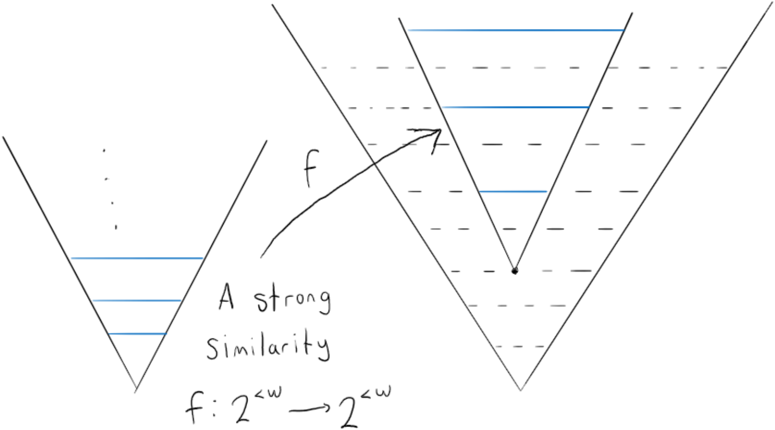
There is a number $T(A) < \omega$ so that for every $\ell < \omega$ and any coloring $\gamma\colon \binom{R}{A} \to \ell$, there is $X \subseteq R$ inducing a copy of the Rado graph so that $\binom{X}{A}$ receives at most $T(A)$ of the colors.

The idea comes from revisiting the binary tree.

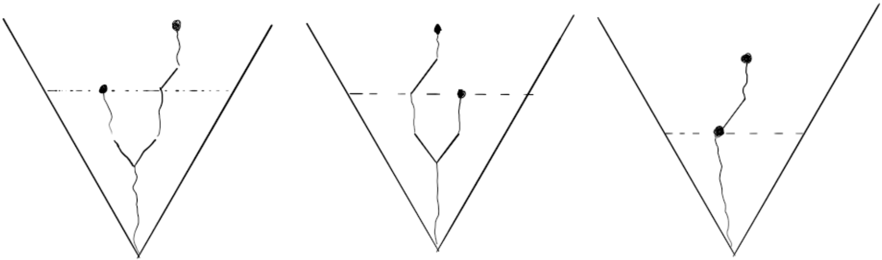
Given $s, t \in 2^{<\omega}$ with $|s| < |t|$, we put $E(s, t)$ iff $t(|s|) = 1$. The resulting graph is bi-embeddable R.

Now every finite subgraph $S \subseteq 2^{<\omega}$ gives rise to an envelope, i.e. the subtree generated by S.

Upper bounds now follow from Milliken's theorem, a difficult Ramsey-like theorem about coloring the strong similarities of the tree $2^{<N}$ into $2^{<\omega}$


f
A strong
similarity
f: 2^{<ω} ⟶ 2^{<ω}

The possible envelopes of two nodes which code an edge:



Soon after, Laflamme, Vuksanović, and Sauer (2007) characterized the value of $T(A)$ exactly for every finite graph A.

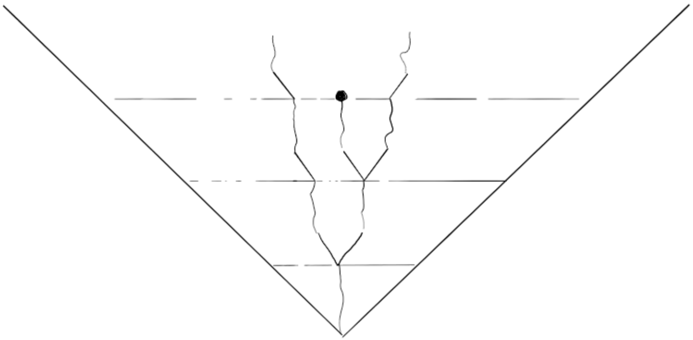
This is done by finding a copy of the Rado graph inside $2^{<\omega}$ giving rise to as few distinct envelopes as possible. We briefly describe the unavoidable envelopes.

One can find $R' \subseteq 2^{<\omega}$ a copy of the Rado graph with the following properties:

1. $R' \subseteq 2^{<\omega}$ is an antichain.
2. For every $m < \omega$, the level 2^m contains at most one node of the form $s \wedge t$ with $s, t \in R'$ (even allowing $s = t$).
3. If $s \neq t \in R'$ with $|s \wedge t| = m$ and $u \in R'$ satisfies $|u| > m$ and $u|_m \neq s \wedge t$, then $u(m) = 0$.

Conversely, any envelope with these three properties must appear in any copy of the Rado graph in $2^{<\omega}$.

An example of a Laflamme-Sauer-Vuksanović envelope:



We can generalize this entire discussion to other Fraïssé structures. Recall that a Fraïssé class is a class $\mathcal{K}$ of finite structures satisfying the following three properties:

1. Hereditary Property (HP): $\mathcal{K}$ is closed under substructures.
2. Joint Embedding Property (JEP): Given $\mathrm{A}, \mathrm{B} \in \mathcal{K}$, there is $\mathcal{C} \in \mathcal{K}$ which embeds both A and B.
3. Amalgamation Property (AP): Given A, B, C $\in \mathcal{K}$ and embeddings $f \colon \mathrm{A} \to \mathrm{B}$ and $g \colon \mathrm{A} \to \mathrm{C}$, there is $\mathrm{D} \in \mathcal{K}$ and embeddings $r \colon \mathrm{B} \to \mathrm{D}$ and $s \colon \mathrm{C} \to \mathrm{D}$ with $r \circ f = s \circ g$.

Given a Fraïssé class $\mathcal{K}$, the Fraïssé limit of $\mathcal{K}$ is a countably infinite structure $\mathbf{K}$ so that:

1. The finite substructures of $\mathbf{K}$, up to isomorphism, are exactly the members of $\mathcal{K}$.
2. If $\mathbf{A} \subseteq \mathbf{K}$ is a finite substructure and $f: \mathbf{A} \to \mathbf{K}$ is an embedding, there is $g \in \mathrm{Aut}(\mathbf{K})$ with $g|_{\mathbf{A}} = f$.

These two properties define $\mathbf{K}$ uniquely up to isomorphism. Write $\mathrm{Flim}(\mathcal{K})$ for this structure.

In this talk, our Fraïssé classes and structures will always satisfy the following extra assumptions:

1. We only consider finite relational languages, and all relational symbols have arity at most 2. We call these finite binary languages. L always denotes such a language.

2. We consider Fraïssé classes $\mathcal{K}$ which have free amalgamation:

   Given an amalgamation problem $(f\colon \mathrm{A} \to \mathrm{B};\, g\colon \mathrm{A} \to \mathrm{C})$, we can find a solution $(r\colon \mathrm{B} \to \mathrm{D};\, s\colon \mathrm{C} \to \mathrm{D})$ with

   - $\mathrm{Im}(r) \cap \mathrm{Im}(s) = \mathrm{Im}(r \circ f) = \mathrm{Im}(s \circ g)$
   - Whenever R is a relational symbol, $x, y \in D$, and $R^D(x, y)$ holds, we have $x, y \in \mathrm{Im}(r)$ or $x, y \in \mathrm{Im}(s)$

Definition

Let K be a countably infinite first-order structure, and let A be a finite structure with $\mathrm{Emb}(\mathsf{A},\mathsf{K}) \neq \emptyset$. Let $\ell < r < \omega$. We write

$$\mathsf{K} \to (\mathsf{K})^{\mathsf{A}}_{r,\ell}$$

if for any coloring $\gamma\colon \mathrm{Emb}(\mathsf{A},\mathsf{K}) \to r$, there is $\eta \in \mathrm{Emb}(\mathsf{K},\mathsf{K})$ with $|\gamma[\eta \cdot \mathrm{Emb}(\mathsf{A},\mathsf{K})]| = |\mathrm{Im}(\gamma \cdot \eta)| \leq \ell$.

The Ramsey degree of A in K is the least $\ell < \omega$, if it exists, with $\mathsf{K} \to (\mathsf{K})^{\mathsf{A}}_{r,\ell}$ for every $r > \ell$.

If $\mathcal{K}$ is a Fraïssé class with limit K, we say that $\mathsf{A} \in \mathcal{K}$ has big Ramsey degree $\ell < \omega$ if A has Ramsey degree ℓ in K.

We say that $\mathcal{K}$ has finite big Ramsey degrees if every $\mathsf{A} \in \mathcal{K}$ has some finite big Ramsey degree.

This definition colors embeddings instead of substructures, but one can easily move between the two notions. The "embedding" big Ramsey degree of $\mathrm{A} \in \mathcal{K}$ is just the "substructure" big Ramsey degree of $\mathrm{A} \in \mathcal{K}$ times $|\mathrm{Aut}(\mathrm{A})|$.

So we can rephrase the result of Pouzet and Sauer to say that the edge has big Ramsey degree 4 in the class of finite graphs.

More difficult example: Henson's triangle-free graph K, the Fraïssé limit of the class $\mathcal{K}$ of finite, triangle-free graphs.

Theorem (Dobrinen (2020, appearing 2016))

The class $\mathcal{K}$ of finite triangle-free graphs has finite big Ramsey degrees.

Let's consider the binary tree again to see why this theorem is so much harder than Sauer's result...

Problem 1: the graph structure we placed on $2^{<\omega}$ is not bi-embeddable with K.

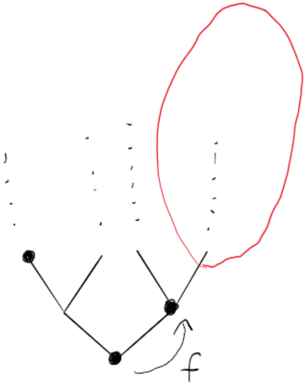
Solution: designate some of the vertices of $2^{<\omega}$ as coding nodes.

One way to do this is to revisit the Erdős-Hajnal-Pósa strategy of associating the vertices of an enumerated graph with nodes of a binary tree. The exact same mapping makes sense for any countable graph.

$\mathrm{K} = \{v_n : n < \omega\}$, and set $s_n \in 2^n$ via $s_n(m) = 1$ iff $E(v_m, v_n)$.

Problem 2: now $2^{<\omega}$ has a whole bunch of coding nodes, making tree embeddings more complicated. We have to send coding nodes to coding nodes.

But it's worse that that...we have to anticipate ahead of time where coding nodes can appear, and how different coding nodes in different parts of the binary tree will interact with each other.



There are no coding nodes extending 11.

Setting $f(\emptyset) = 1$ is "locally" OK, but can't define $f(1)$

Towards the first main result of today's talk:

A relational structure is called irreducible if it is not a free amalgam of any two proper substructures.

If $\mathcal{F}$ is a set of finite, irreducible L-structures, then $\mathrm{Forb}(\mathcal{F})$, the class of finite L-structures which do not embed any structure from $\mathcal{F}$, is a free amalgamation class.

Conversely, every Fraïssé free amalgamation class has the form $\mathrm{Forb}(\mathcal{F})$ for some set of finite irreducible structures.

$\mathcal{F}$ could be infinite.

Theorem (Z. 2020)

Let L be a finite binary language, and suppose $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ for $\mathcal{F}$ a finite set of finite irreducible L-structures. Then $\mathcal{K}$ has finite big Ramsey degrees.

Sauer (2002): there are free amalgamation classes of finite directed graphs where vertices do not have finite big Ramsey degree.

Proof idea (using triangle-free graphs as our running example):

Revisit the binary tree with coding nodes. Add extra structure to tell us how to anticipate coding nodes.

Consider $2^m \subseteq 2^{<\omega}$. Each $t \in 2^m$ represents a type over $\{v_0, \ldots, v_{m-1}\}$.

Given $X \subseteq 2^m$, an X-labeled graph is a graph B along with a map $\gamma^B \colon B \to X$.

$B[m]$ is the graph on $B \sqcup \{v_0, \ldots, v_{m-1}\}$ with edges between $y \in B$ and $\{v_0, \ldots, v_{m-1}\}$ described by the type $\gamma^B(y)$.

$\mathcal{K}(X) = \{\text{finite } X\text{-labeled } B : B[m] \text{ is triangle-free}\}$

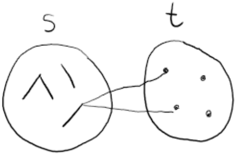
We equip every level subset $X \subseteq 2^{<\omega}$ with the information $\mathcal{K}(X)$. The result is an aged coding tree.

For triangle-free graphs, this extra structure on level sets of $2^{<\omega}$ is entirely described by what happens on pairs of nodes.

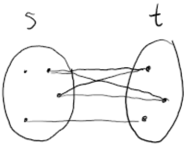
The next illustration only considers pairs of nodes where each node can extend to a coding node.

Possible ages of pairs of nodes in $2^{<\omega}$

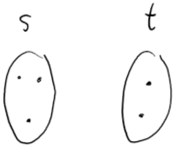
(triangle-free graphs)



No edges where both vertices have label t.



Any edge must connect one vertex of each label.



No edges

One then considers aged embeddings from finite coding trees into $2^{<\omega}$, i.e. strong similarities that respect coding nodes and the age-set structure.

A Ramsey-like theorem similar to Milliken's theorem can then be proven for these objects. The proof uses forcing, ultrafilters, the Erdős-Rado Theorem, and other fun stuff.

Unlike the case for all finite graphs, we don't immediately get finite upper bounds on the big Ramsey degree; we have to construct a subcopy of K so that for a given finite triangle-free graph, copies of this finite subgraph only generate finitely many shapes of envelope. The problem is the coding nodes.

Recently, Dobrinen, and independently Balko, Chodounský, Hubička, Konečný, Vena, and Z. have given a precise characterization of the big Ramsey degrees of triangle free graphs.

We then joined forces to prove:

Theorem (BCDHKVZ (2020))

If $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ for $\mathcal{F}$ a finite set of finite irreducible L-structures, then we can exactly characterize the big Ramsey degrees.

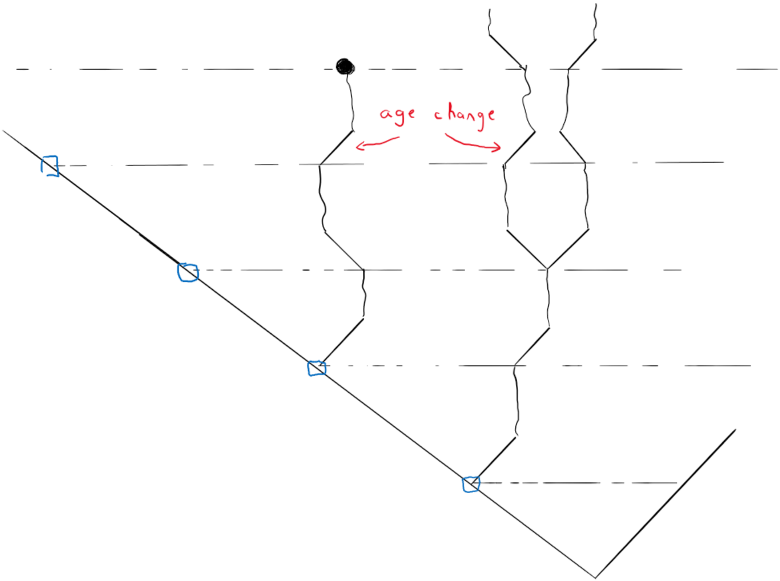
Furthermore, if $\mathsf{K} = \mathrm{Flim}(\mathcal{K})$, then K admits a big Ramsey structure.

There is a subset of coding nodes $K' \subseteq 2^{<\omega}$ coding the Henson triangle-free graph so that the following all hold:

1. $K' \subseteq 2^{<\omega}$ is an antichain.
2. All "external" coding nodes (the coding nodes not in K', but that we need to make envelopes) are 0^n for some $n < \omega$.
3. Any non-coding level of the envelope has either exactly one splitting node or exactly one age change between two nodes.
4. If m is a non-coding level and $t \in K'$ is such that $t|_m$ isn't a splitting node or involved in an age change, then $t(m) = 0$.
5. If $t \in K'$ with $|t| = m$ and $x \in K'$ with $x|_m \notin \{0^m, t\}$, then $\mathcal{K}(x|_m, t)$ is the set of edgeless graphs iff $x(m) = 0$.

Conversely, any envelope with these properties must appear in any set of coding nodes coding the Henson triangle-free graph.

An example of a BCDHKVZ envelope:



Thanks!