The strength of Borel Wadge comparability

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20^{th} April 2021

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Wadge comparability

Wadge reducibility

We work in Baire space ω^ω.

Definition

Let $A, B \subseteq \omega^\omega$. We say that A is *Wadge reducible* to B (and write $A \leqslant_W B$) if A is a continuous pre-image of B: for some continuous function $f \colon \omega^\omega \to \omega^\omega$,

$$x \in A \Leftrightarrow f(x) \in B.$$

This gives rise to Wadge equivalence and Wadge degrees.

Wadge comparability

The Wadge degrees of Borel sets are almost a linear ordering:

Theorem (Wadge comparability, c. 1972)

For any two Borel sets A and B, either

- $A \leqslant_W B$, *or*
- $B \leqslant_W A^\complement$.

Further facts on Wadge degrees of Borel sets:

- They are well-founded (Martin and Monk);
- They alternate between self-dual and non self-dual degrees;
- The rank of the $\mathbf{\Delta}^0_2$ sets is ω_1, other ranks given by base-ω_1 Veblen ordinals.

The Wadge game

Wadge comparability is usually proved by applying determinacy to the game $G(A, B)$:

- Player I chooses $x \in \omega^\omega$;
- Player II chooses $y \in \omega^\omega$;
- Player II wins iff $x \in A \Leftrightarrow y \in B$.

A winning strategy for Player II gives a Wadge reduction of A to B; a winning strategy for player I gives a Wadge reduction of B to $A^\complement$.

Hence, AD implies Wadge comparability of all sets.

Wadge comparability and determinacy

- $\mathbf{\Pi}^1_1$ determinacy is equivalent to Wadge comparability of $\mathbf{\Pi}^1_1$ sets (Harrington 1978);
- $\mathbf{\Pi}^1_2$ determinacy is equivalent to Wadge comparability of $\mathbf{\Pi}^1_2$ sets (Hjorth 1996).

Borel determinacy is provable in ZFC (Martin 1975) and so Wadge comparability of Borel sets is provable in ZFC.

Theorem (H.Friedman 1971)

Borel determinacy requires ω_1 iterations of the power set of $\mathbb{N}$.

In particularly, Borel determinacy is not provable in Z_2.

The strength of Borel Wadge comparability

Theorem (Louveau and Saint Raymond, 1987)

Borel Wadge comparability is provable in Z_2.

Theorem (Loureiro, 2015)

- *Lipschitz comparability for clopen sets is equivalent to* ATR_0.
- *Wadge comparability for some Boolean combinations of open sets is provable in* Π^1_1-CA_0.

Theorem

Borel Wadge comparability is provable in $\mathsf{ATR}_0 + \Pi^1_1$*-induction.*

Background: Effective methods in DST

Boldface and lightface

Effective descriptive set theory relies on the relationship between lightface and boldface pointclasses:

- A set is open $\Leftrightarrow$ it is $\Sigma^0_1(x)$ for some parameter x;
- A function is continuous $\Leftrightarrow$ it is x-computable for some x;
- A set is Borel $\Leftrightarrow$ it is hyperarithmetic in some x;

and so on.

Example

Theorem (Luzin/Suslin)

If B is Borel, f is continuous and $f \restriction B$ is 1-1, then $f[B]$ is Borel.

Proof.

Wlog, f is computable and B is Δ^1_1.

Let $x \in B$; let $y = f(x)$. Then x is the unique solution of

$$x \in B \ \& \ y = f(x),$$

so x is a $\Delta^1_1(y)$-singleton; it follows that $x \in \Delta^1_1(y)$.

So

$$y \in f[B] \iff (\exists x)\ y = f(x) \iff (\exists x \in \Delta^1_1(y))\ y = f(x).$$

The second condition is Σ^1_1; by Spector-Gandy, the third is Π^1_1. $\square$

Other examples

There are many other examples:

- Measurability of $\mathbf{\Pi}^1_1$ sets (Sacks);
- Perfect set property of $\mathbf{\Sigma}^1_1$ sets;
- $\mathbf{\Pi}^1_1$ uniformisation (Kondo, Addison);
- Louveau (1980) used his separation theorem to solve the section problem for Borel classes;
- E_0 dichotomy for Borel equivalence relations: Harrington,Kechris,Louveau (1990);
- $\mathbb{G}_0$ dichotomy for Borel chromatic numbers: Kechris,Solecki,Todorcevic (1999).

Generalised homeomorphisms and the Turing jump

Generalised homeomorphisms

Proposition (Kuratowski)

A set A is $\mathbf{\Delta}^0_{1+\alpha}$ iff there are:

- *a closed set E;*
- *a clopen set D; and*
- *a bijection $h\colon \omega^\omega \to E$ such that:*
  - *h is Baire class α;*
  - *h^{-1} is continuous*

such that for all x,

$$x \in A \Leftrightarrow h(x) \in D.$$

Lightface version:

Proposition

A set A is $\Delta^0_{1+\alpha}$ iff there is a Δ^0_1 set D such that for all x,

$$x \in A \Leftrightarrow x^{(\alpha)} \in D.$$

Making Borel sets clopen

Proposition

If A is Borel, then there is a Polish topology on ω^ω extending the standard one, which has the same Borel sets, and in which A is clopen.

Proof.

Pull back the topology from the image of the α-jump. $\square$

True stages

Iterated priority arguments

Theorem (Watnick;Ash,Jockusch,Knight)

Let α be a computable ordinal, and let L be a $\Delta^0_{2\alpha+1}$ linear ordering. Then $\mathbb{Z}^\alpha \cdot L$ has a computable copy.

This is usually presented as an application of Ash's "η-systems". His metatheorem is used to conduct priority arguments at level $\varnothing^{(\eta)}$.

Other methods:

- Harrington "worker arguments".
- Lempp-Lerman trees of strategies.

1-true stages

Montalbán gave a dynamic presentation of Ash's metatheorem. His technique of *α-true stages* allows for very fine control of the priority argument at each level $\beta \leqslant \alpha$.

For $\alpha = 1$ this was done by Lachlan. The main idea:

- Suppose that $\langle A_s \rangle$ is a computable enumeration of a c.e. set $A \subseteq \mathbb{N}$. Say that at stage s, a single number n_s enters A. A stage s is a *Dekker nondeficiency stage* if for all $t \geqslant s$, $n_t \geqslant n_s$. There are infinitely many nondeficiency stages. (This is used to show that every nonzero c.e. degree contains a simple set.)
- Lachlan: Suppose that at stage s, we guess that $A_s \restriction n_s$ is an initial segment of A. Then at nondeficiency stages the guess is correct.

A stage s is *1-true* if $\varnothing'_s \restriction n_s \prec \varnothing'$.

Finite injury arguments

Suppose that we want to perform a finite injury priority construction. We construct some computable object, but we really want to know some Δ^0_2 information to do so. At each stage, $\varnothing'_s$ gives us answers to some of our questions.

- We do not know which stages are 1-true.
- But from the point of view of a stage t, looking back:

  If $s < t$ is 1-true, then t thinks that s is 1-true.
  If $s < t$ is not 1-true, then t may not have enough information to know it. However:
  If t is 1-true, then $s < t$ is 1-true iff s is 1-true.

The relation "s appears 1-true at stage t" (denoted by $s \leqslant_1 t$) is computable for finite stages s and t. This is what allows us to perform a computable construction.

α-true stages

Montalbán's idea was to iterate this up the hyperarithmetic hierarchy.

- A stage is 2-true if it is 1-true relative to the 1-true stages. Similarly, $s \leqslant_2 t$ if $s \leqslant_1 t$, and further, looking at the enumeration of $\varnothing''$ using the oracles $\varnothing'_r$ for $r \leqslant_1 t$, we have not yet discovered that s is a deficiency stage for that enumeration.
- Similarly for $n + 1$.
- For limit λ, s is λ-true if it is β-true for all $\beta < \lambda$, and similarly for $s \leqslant_\lambda t$. This mirrors $\varnothing^{(\lambda)} = \bigoplus_{\beta<\lambda} \varnothing^{(\beta)}$.
- Main question: why are there λ-true stages? Some modificaton using a diagnoal intersection is needed.
- Technical device: we can replace $\varnothing^{(\beta)}$ by the sequence of β-true stages.

Relativised α-true stages

The construction of α-true stages can be uniformly relativised to oracles $x \in \omega^\omega$. The notion $s \leqslant_\alpha t$ relative to x can be made to only depend on $x \upharpoonright t$. We obtain relations $\leqslant_\alpha$ on $\omega^{\leqslant\omega}$ with a variety of nice properties:

- $\sigma \leqslant_0 \tau \Leftrightarrow \sigma \preccurlyeq \tau$.
- For each β, the relation $J_\beta = (\omega^{<\omega}; \leqslant_\beta)$ is a computable tree.
- $$x \mapsto \langle \sigma \,:\, \sigma \prec_\beta x \rangle$$
  is a bijection between ω^ω and the paths of J_β.
- The relation $\sigma \prec_\beta x$ is $\Delta^0_{1+\beta}$.
- A set A is $\Sigma^0_{1+\beta}$ iff there is a c.e. set $U \subseteq \omega^{<\omega}$ such that
  $$x \in A \Leftrightarrow (\exists \sigma \prec_\beta x)\sigma \in U.$$
- The relations $\leqslant_\beta$ are nested and continuous.

Some simple applications

Change of topology

Proposition

Let $\beta < \omega_1^{\mathrm{ck}}$. There is a Polish topology on ω^ω extending the standard one such that:

- *Every standard $\Delta^0_{1+\beta}$ set is clopen in the new topology;*
- *Every new open set is old $\mathbf{\Sigma}^0_{1+\beta}$.*

Proof.

Define the distance between x and y to be $2^{-|\sigma|}$, where σ is greatest such that $\sigma \prec_\beta x$ and $\sigma \prec_\beta y$. □

Hausdorff-Kuratowski

Theorem (Haudorff-Kuratowski)

For each countable ξ,

$$\mathbf{\Delta}^0_{\xi+1} = \bigcup_{\eta<\omega_1} D_\eta(\mathbf{\Sigma}^0_\xi)$$

Where $D_\eta(\mathbf{\Sigma}^0_\xi)$ is the η^{th} level of the Hausdorff difference hierarchy: sets of the form

$$\bigcup \left(A_i - \bigcup_{j<i} A_j\right) \quad [\![i < \eta \;\&\; \text{parity}(i) \neq \text{parity}(\eta)]\!]$$

where $A_0 \subseteq A_1 \subseteq \cdots$ is an increasing η-sequence of $\mathbf{\Sigma}^0_\xi$ sets.

Shoenfield, Hausdorff, and Ershov

Uniform limit lemma:

- A set A is Δ^0_2 iff there is a computable function $f\colon \omega^{<\omega} \to \{0,1\}$ such that for all x,

$$A(x) = \lim_{\sigma \prec x} f(\sigma).$$

A set A is $D_\eta(\Sigma^0_1)$ iff the relation $x \in A$ is η-c.e., uniformly in x:
There are computable functions $f\colon \omega^{<\omega} \to \{0,1\}$ and $r\colon \omega^{<\omega} \to \eta+1$ such that:

- For all x, $A(x) = \lim_{\sigma \prec x} f(\sigma)$;
- If $\sigma \preccurlyeq \tau$ then $r(\tau) \leqslant r(\sigma)$;
- If $r(\sigma) = \eta$ then $f(\sigma) = 0$;
- If $\sigma \preccurlyeq \tau$ and $f(\sigma) \neq f(\tau)$ then $r(\tau) < r(\sigma)$.

Effective Hausdorff-Kuratowski

Theorem (Louveau and Saint Raymond,1988; Selivanov 2003; Pauly 2015)

$\Delta^0_2 = \bigcup_{\eta < \omega_1^{ck}} D_\eta(\Sigma^0_1)$.

Proof.

Suppose that A is Δ^0_2; fix a computable approxmation $f\colon \omega^{<\omega} \to \{0,1\}$ for A.

Set:

- $r(\sigma) = 0$ if $(\forall \tau \succcurlyeq \sigma)\, f(\tau) = f(\sigma)$.
- $r(\sigma) \leqslant \gamma$ if for all $\tau \succ \sigma$, if $f(\tau) \neq f(\sigma)$ then $r(\tau) < \gamma$.

The empty string is ranked, otherwise we construct a path on which $f(\sigma)$ does not converge.

The ranking process is hyperarithmetical (need one jump for each level), so the rank of the empty string is computable.

Then $\langle f(\sigma), r(\sigma)\rangle_{\sigma \prec x}$ is as required. □

Effective Hausdorff-Kuratowski

Theorem

For any computable ξ,

$$\Delta^0_{\xi+1} = \bigcup_{\eta<\omega_1^{\mathsf{ck}}} D_\eta(\Sigma^0_\xi).$$

Proof.

Repeat the previous proof, but replace $\sigma \preccurlyeq \tau$ by $\sigma \preccurlyeq_\xi \tau$. □

Wadge analysis of Δ classes

Let λ be a limit ordinal, and let Γ be a pointclass.

$$\mathrm{PU}_{<\lambda}(\Gamma)$$

is the pointclass of all sets A for which there is some $\alpha < \lambda$ and a partition (C_n) of ω^ω of $\mathbf{\Delta}^0_\alpha$ sets such that for all n, $A \restriction C_n \in \Gamma$.

Theorem (Wadge)

For every limit $\lambda < \omega_1$,

$$\mathbf{\Delta}^0_\lambda = \mathrm{PU}^{(\omega_1)}_{<\lambda}(\mathbf{\Delta}^0_{<\lambda}).$$

Effective Wadge

Theorem

For every limit $\lambda < \omega_1^{\mathrm{ck}}$,

$$\Delta^0_\lambda = \mathrm{PU}^{(\omega_1^{\mathrm{ck}})}_{<\lambda}(\Delta^0_{<\lambda}).$$

Let $A \in \Delta^0_\lambda$. There is a clopen set D such that

$$x \in A \Leftrightarrow x^{(\lambda)} \in D.$$

The tree of $\sigma \in J_\lambda$ which decide D is computable and well-founded, so has computable rank.

Fact:

- The relation $\sigma \prec_\lambda x$ is $\Delta^0_{\lambda_n}$, where n is the height of σ in J_λ.

Hence, by induction on $\alpha = \mathrm{rk}(\sigma)$,

$$A \restriction \{x : \sigma \prec_\lambda x\} \in \mathrm{PU}^{(\alpha)}(\Delta^0_{<\lambda}).$$

Wadge comparability

Describing classes, using determinacy

There are comprehensive descriptions of Borel Wadge classes:

- Louveau (1983);
- Duparc (2001);
- Selivanov for k-partitions (2007,2017);
- Kihara and Montalbán for functions into a countable BQO (2019).

To show every class is covered, one usually:

- Uses determinacy to show the degrees are almost well-ordered;
- In some form, perform induction on the Wadge degrees to show each one is described.

This route is closed to us.

The structure of the argument

We follow Louveau and Saint Raymond:

- Define a collection of descriptions for (non self-dual) Wadge classes.
- Show these all have universal sets.
- Show that the described classes are almost linearly-ordered.
- Show that the described classes are well-founded.
- Perform a careful analysis of the ambiguous part $\Delta(\Gamma)$ for each described class Γ, to conclude that every class is described.

In second-order arithmetic, we need to do everything effectively.

The main step

The main step is the following separation result.

Theorem (Louveau and Saint Raymond)

Suppose that Γ is a described class. Let $A \in \Gamma$; let B_0 and B_1 be two disjoint Σ^1_1 sets. Then either:

1. *There is a continuous reduction of $(A, A^\complement)$ into (B_0, B_1); or*
2. *There is a $\check{\Gamma}$ separator of B_0 from B_1.*

As a result: if A is universal for Γ, and B is Borel, then either $A \leqslant_W B$, or $B \in \check{\Gamma}$, in which case $B \leqslant_W A^\complement$.

This shows that the described classes are almost linearly ordered.

Unravelling games

The direct way to prove this result would be to use determinacy for a naturally associated game. However, Louveau and Saint Raymond show:
To each class Γ we can associate a *closed* game $G(A, B_0, B_1)$ for which:

- A winning strategy for II gives a continuous reduction of $(A, A^\complement)$ to (B_0, B_1);
- From a winning strategy for I we can find a $\check{\Gamma}$ separator of B_0 from B_1.

Our main step is to give a relatively simple description of such a game. Take for example the class $\Gamma = \Sigma^0_\xi$. Suppose that T_i is a tree whose projection is B_i.

- Player I plays $x \in A$ or in $A^\complement$.
- Player II attempts to play $y \in B_0$ or B_1 and a witness $f \in [T_i]$.
- The bits of y are read off the ξ-true stages of II's play.

Thank you