Computability-theoretic analogs of combinatorial cardinal characteristics

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Set theory

- Cardinal characteristics (of the continuum) are natural cardinals that measure the deviation from CH.
- Many are based on binary relations. E.g. $\mathfrak{b}$ is the unbounding number: the least size of a class of functions on ω that is not dominated by a single function.
- Others are based on cardinals of subclasses of $[\omega]^\omega$ (the infinite subsets of ω) viewed up to almost equality.
- One of them is called the almost disjointness number, denoted $\mathfrak{a}$.
- This is the least size of a maximal almost disjoint family of subsets of ω.
- Almost disjoint means that any two distinct sets in the family have finite intersection.

ZFC relations, and inequalities

In this area of set theory, one tries to obtain ZFC relations between cardinal characteristics. Recall:

- $\mathfrak{b}$ is the unbounding number: the least size of a class of functions on ω that is not dominated by a single function.
- $\mathfrak{a}$ is the least size of a maximal almost disjoint family of subsets of ω.

Fact

$\mathfrak{b} \leq \mathfrak{a}$.

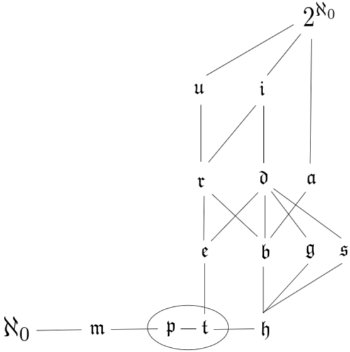
(See e.g. Logic Blog '19 for a proof of this well known fact.)
In the opposite direction, using (iterated) forcing one tries to separate cardinal characteristics.

Ultrafilter, tower, and independence number

Further cardinal characteristics based on properties of subsets of $[\omega]^\omega$ under almost inclusion $\subseteq^*$:

- the ultrafilter number $\mathfrak{u}$ is the least size of a set with upward closure a free ultrafilter on ω,
- the tower number $\mathfrak{t}$ is the least size of a linearly ordered subset of $[\omega]^\omega$ that can't be extended by putting a new element below all given elements,
- the independence number $\mathfrak{i}$ is the least size of a maximal independent set in the Boolean algebra $\mathcal{P}(\omega)/=^*$.

Diagram of ZFC relations (Soukup, 2018)



$\mathfrak{r}$ and $\mathfrak{s}$ are the unreaping and splitting numbers, respectively. Their analogs in computability have been studied e.g. Brendle et al, '14.

$\mathfrak{e}$ is the escaping number due to Brendle and Shelah. Its analog in computability theory has been studied by Valverde and Tveite (2017).

Collections of computable subsets of ω

- Our basic objects will be collections of infinite computable sets in the context of almost inclusion.
- Such a collection $\mathbf{C}$ is encoded by a set F such that
$$\mathbf{C} = \mathbf{C_F} = \{F^{[n]} \colon n \in \mathbb{N}\}.$$

Definition

$F^{[n]}$ denotes the column $\{x \colon \langle x, n \rangle \in F\}$ of a set $F \subseteq \omega$.
We will usually denote this by F_n.

Analogous mass problems in computability

A mass problem is a set of functions $f\colon \mathbb{N} \to \mathbb{N}$.

- We view properties of such encoded collections of computable sets as mass problems. They consist of the characteristic functions of the encoding sets F.
- One can compare their complexities via Muchnik reducibility $\leq_w$ and the stronger, uniform Medvedev reducibility $\leq_s$:

$$\mathcal{C} \leq_s \mathcal{D} \text{ if } \exists \Phi \forall X \in \mathcal{D} [\Phi^X \in \mathcal{C}].$$

- ZFC relations of cardinal charactertistics correspond to Muchnik/ Medvedev reductions of their analogs?

The mass problems $\mathcal{A}$ and $\mathcal{T}$

of maxinally almost disjoint sets

and maximal towers

The mass problem $\mathcal{A}$ of MAD sets

We will often identify a set $F \subseteq \mathbb{N}$ and the collection $\mathbf{C}_F = \{F_n : n \in \mathbb{N}\}$ of computable sets described by F.

We say that $F \subseteq \mathbb{N}$ is a almost disjoint, AD in brief, if

each F_n is infinite, computable, and $F_n \cap F_k =^* \emptyset$ for $n \neq k$.

Definition (Analog of almost disjointness number)

- The mass problem $\mathcal{A}$ is the class of sets F such that $\mathbf{C}_F$ is maximal almost disjoint (MAD) for the computable sets.
- Namely, $\mathbf{C}_F$ is AD, and for each infinite computable set R, there is n such that $R \cap F_n$ is infinite.

No MAD set F is computable

Proposition

No MAD set F is computable.

Suppose F is almost disjoint and computable.

Let $r_{-1} = 0$, and r_n be the least number $r > r_{n-1}$ such that $r \in F_n - \bigcup_{i<n} F_i$.

Then the computable set $R = \{r_0, r_1, \ldots\}$ shows that F is not maximally almost disjoint.

The mass problem $\mathcal{T}$ of maximal towers

We say that $G \subseteq \mathbb{N}$ is a tower (or $\mathbf{C}_G$ is a tower) if G_n is computable for each n, and

$G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite.

Definition (Analog of tower number $\mathfrak{t}$)

- The mass problem $\mathcal{T}$ is the class of sets G such that $\mathbf{C}_G$ is a tower that is maximal in the computable sets.
- Namely, for each infinite computable set R there is n such that $R - G_n$ is infinite.

$\mathcal{A}$ and $\mathcal{T}$ are Medvedev equivalent

$\boxed{\mathcal{A} \leq_s \mathcal{T}:}$ Define a Turing functional Diff by letting

$$\mathrm{Diff}(G) = \text{the set } F \text{ such that } F_n = G_n - G_{n+1} \text{ for each } n.$$

If G is a maximal tower then $F = \mathrm{Diff}(G)$ is MAD. For, if R is infinite computable then $R - G_n$ is infinite for some n, and hence $R \cap F_i$ is infinite for some $i < n$.

$\boxed{\mathcal{T} \leq_s \mathcal{A}:}$ Define a Turing functional Cp by

$$\mathrm{Cp}(F) = \text{ the set } G \text{ such that } G_n = \mathbb{N} - \bigcup_{i<n} F_n \text{ for each } n.$$

If F is AD then G is a tower, and if F is MAD then G is a maximal tower.

Non-low oracles uniformly compute a set in $\mathcal{T}$

Theorem

$\mathcal{T}$ (the mass problem of maximal towers) $\leq_s$ NonLow.

Proof. Let x, y, z denote binary strings; we identify x with the number $1x$ via the binary expansion. Define a Turing functional Φ for the Medvedev reduction: $\Phi^Z = G$, where for each n

$$G_n = \{x \colon\ n \leq s := |x| \wedge Z'_s \upharpoonright n = x \upharpoonright n\}.$$

- For each n we have $G_{n+1} \subseteq^* G_n$ and $G_n - G_{n+1}$ is infinite.
- Each G_n is computable since for large enough s the string $Z'_s \upharpoonright n$ has settled.

If $R \subseteq^* G_n$ for each n, where R is an infinite set, then $Z'(k) = \lim_{x \in R, |x| > k} x(k)$, and hence $Z' \leq_T R'$. So if $Z \in \text{NonLow}$ then such an R cannot be computable. Hence $\Phi^Z \in \mathcal{T}$.

C.e. MAD set by a finitary priority construction

Theorem

For each noncomputable c.e. set A, there is c.e. MAD set $F \leq_T A$.

Let $V_{2e} = W_e$ and $V_{2e+1} = \mathbb{N}$ for each e. Build an auxiliary c.e. set $S \leq_{\mathrm{T}} A$. Then let $F \leq_T A$ be defined by $F_e = S_{2e} \cup S_{2e+1}$.

$$P_n\colon V_e - \bigcup_{i<n} S_i \text{ infinite } \Rightarrow |S_e \cap V_e| \geq k \qquad (n = \langle e, k \rangle).$$

At stage s we say that P_n is satisfied if $|S_{e,s} \cap V_{e,s}| \geq k$.

Construction.

Stage $s > 0$. For each $n < s$ such that P_n is not satisfied, if there is $x \in V_{e,s} - \bigcup_{i<n} S_{i,s}$ such that $x > \max(S_{e,s-1})$, $x \geq 2n$ and $A_s \upharpoonright x \neq A_{s-1} \upharpoonright x$, then put $\langle x, e \rangle$ into S (i.e., put x into S_e).

Index guessable oracles

Indices for columns of a MAD hard to compute

A characteristic index for a set M is an e such that $\chi_M = \varphi_e$.

Proposition

Suppose F is maximally almost disjoint. Then $\emptyset'$ is not able to compute, from input n, a characteristic index for F_n.

Proof.

Assume otherwise. Then there is a computable function f such that $\varphi_{\lim_s f(n,s)}$ is the characteristic function of F_n.

Let $\widehat{F}$ be defined as follows. Given n, x , compute the least $s > x$ such that $\varphi_{f(n,s),s}(x)\downarrow$. If the value is not 0 put x into $\widehat{F}_n$.

Clearly $\widehat{F}$ is computable. Since $F_n =^* \widehat{F}_n$ for each n, the set $\widehat{F}$ is MAD, contradicting the fact obtained above. $\square$

Index guessable oracles

Definition

We call an oracle L index guessable if whenever Φ_e^L is computable then $\emptyset'$ can compute from e an index for its characteristic function.

In other words, there is a functional Γ a such that

$$\Phi_e^L \text{ is computable} \Rightarrow \Gamma(\emptyset'; e) \text{ is an index for it, i.e., } \Phi_e^L = \varphi_{\Gamma(\emptyset'; e)}.$$

- it's easy to give a direct proof that index guessability implies lowness.
- The proposition above (Slide 16) implies that no index guessable set computes a MAD set.
- Thus, by the permitting result above (Slide 14), a c.e., index guessable set is computable.

Definition (Recall)

We call an oracle L index guessable if whenever $F = \Phi_e^L$ is computable then $\emptyset'$ can compute from e an index for its characteristic function.

Proposition

Suppose L is Δ_2^0 and 1-generic. Then L is index guessable.

Proof: one notes that

$$C_e = \{\tau \colon (\exists p)\, \Phi_e^\tau(p) \neq F(p)\}$$

is not dense along L. Since L is Δ_2^0, this can be used to have $\emptyset'$ compute an index for F.

Oracles not computing a MAD

So we have:

1-generic $\Delta^0_2 \Rightarrow$ index guessable $\Rightarrow$ computes no MAD $\Rightarrow$ low.

- $\Delta^0_2 \cap$ 1-generic is downward closed (Haught)
- We only know at present that the last arrow cannot be reversed.
- To see this recall that any noncomputable c.e. set computes a MAD.

The mass problem $\mathcal{U}$,

an analog of the ultrafilter number

Definition (Analog of the ultrafilter number $\mathfrak{u}$)

The mass problem $\mathcal{U}$ consists of the sets F such that each F_n is computable,

- $F_{n+1} \subseteq^* F_n$ and $F_n - F_{n+1}$ is infinite (i.e., F is a tower).
- for each computable set R there is n such that $F_n \subseteq^* R$ or $F_n \subseteq^* \overline{R}$.

We say that F (or, more precisely, $\mathbf{C}_F$) is an ultrafilter base (UFB) within the computable sets.

Fact

$\mathcal{U} \subseteq \mathcal{T}$, that is, each UFB is a maximal tower.
So we trivially have $\mathcal{T} \leq_s \mathcal{U}$ via the identity reduction.

Example of an UFB

- Take any r-maximal set C.
- By definition of r-maximality, the computable sets R such that $R \cup C$ is cofinite form an ultrafilter.
- Using this one can obtain an ultrafilter base $\mathbf{C}_F$ where $F \leq_T \emptyset''$.

Recall that each nonlow computes a maximal tower. So, by the following, not each maximal tower computes an ultrafilter base.

Proposition (to be strengthened)

No ultrafilter base F is computably dominated.

Proof.

Let $g(n)$ be the least number $> n$ in $\bigcap_{i<n} F_i$. Then $g \leq_T F$.
Assume that there is a computable function $p \geq g$. The conditions $n_0 = 1$ and $n_{k+1} = p(n_k)$ define a computable sequence.
So the set

$$E = \bigcup_i [n_{2i}, n_{2i+1})$$

is computable.
Clearly $F_n \not\subseteq^* E$ and $F_n \not\subseteq^* \overline{E}$ for each n. So F is not an ultrafilter base. $\square$

Highness and mass problems

Our aim is to show that the degrees of ultrafilter bases coincide with the high degrees. How do we formulate a version of this for strong reductions?

- Let DomFcn denote the mass problem of functions h that dominate every computable function, and also satisfy $h(s) \geq s$ for all s.
- Let $\mathrm{Tot} = \{e \colon \phi_e \text{ is total}\}$. Note that F is high iff $\mathrm{Tot} \leq_T F'$.
- The approximations to Tot are the $\{0,1\}$-valued binary functions f such that $\lim_s f(e,s) = \mathrm{Tot}(e)$.

Fact (Martin, morally)

DomFcn is Medvedev equivalent to the mass problem of approximations to Tot.

Classifying the complexity of ultrafilter bases

Theorem

The mass problem DomFcn of dominating functions

is Medvedev equivalent to

the mass problem $\mathcal{U}$ of ultrafilter bases.

In particular, the degrees of ultrafilter bases are exactly the high degrees.

Proof of DomFcn $\leq_s \mathcal{U}$

Lemma

There is a uniformly computable sequence $P_0, P_1, \ldots$ of nonempty Π^0_1-classes such that for every e,

- *if ϕ_e is total, then P_e contains a single element, and*
- *if ϕ_e is not total, then P_e contains only bi-immune elements.*

Given an ultrafilter base F we have

$$\phi_e \text{ is total} \iff (\exists i)(\exists n)\; [F_i \setminus [0,n] \text{ is a subset of some } X \in P_e \text{ or its complement}]$$

We use this to uniformly compute from F an approximation to Tot in the sense of the Limit Lemma, and hence a dominating function. See Thm. 3.6 in the CDMTCS preprint for detail.

Proof of DomFcn $\geq_s \mathcal{U}$

Let $\langle \psi_e \rangle_{e \in \mathbb{N}}$ be an effective listing of the $\{0, 1\}$ valued partial computable functions defined on an initial segment of $\mathbb{N}$. Let

$$V_{e,k} = \{x \colon \psi_e(x) = k\}.$$

Let $T = \{0, 1, 2\}^{<\infty}$. For $\alpha \in T$ we enumerate in an increasing fashion a (possibly finite) c.e. set S_α. Enumeration is uniform in α.

Let $S_{\emptyset,s} = [0, s)$. If we have defined (at stage s) the set $S_\alpha = \{r_0 < \ldots < r_k\}$, let $\widetilde{S}_\alpha$ contain the numbers of the form r_{2i}.

- Let $S_{\alpha 2} = \widetilde{S}_\alpha$.
- Let $S_{\alpha k} = \widetilde{S}_\alpha \cap V_{e,k}$ for $k = 0, 1$, $e = |\alpha|$.

Proof of DomFcn $\geq_s \mathcal{U}$ continued

Define a uniform list of Turing functionals Γ_e so that the sequence $\langle \Gamma_e^h(t) \rangle_{t \in \mathbb{N}}$ is nondecreasing, for each e and each oracle function h such that $h(s) \geq s$ for each s. We will let $F_e = \{\Gamma_e^h(t) \colon t \in \mathbb{N}\}$.

Definition of Γ_e. Given an oracle function h, we will write a_s for $\Gamma_e^h(s)$. Let $a_0 = 0$. Suppose $s > 0$ and a_{s-1} has been defined.

Let $\alpha \in T$ be the leftmost string of length e such that there is an $x \in S_{\alpha, h(s)}$ with $x > a_{s-1}$. Choose x least for α and let $a_s = x$. If there is no such α let $a_s = a_{s-1}$.

Verification. Suppose h is a dominating function. Then for each e we have $F_e =^* S_\alpha$, where α is the leftmost string of length e such that S_α is infinite.

Co-c.e. ultrafilter bases

Recall so far we only produced a Δ^0_3 ultrafilter base.

However, a modification of the construction above, along with a technique from a 2001 paper on r-maximal sets by Lempp, N. and Solomon, yields the following.

Theorem

There is a co-c.e. ultrafilter base.

Maximally independent families in computability

Given a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$, for each binary string σ we write

$$F_\sigma = \bigcap_{\sigma(i)=1} F_i \cap \bigcap_{\sigma(i)=0} \overline{F}_i.$$

We call (a set F encoding) such a sequence independent if each set F_σ is infinite.

Definition

The mass problem $\mathcal{I}$ is the class of sets F such that $\langle F_n \rangle_{n\in\mathbb{N}}$ is a family that is maximally independent, namely, it is independent, and for each computable set R, there is σ such that $F_\sigma \subseteq^* R$ or $F_\sigma \subseteq^* \overline{R}$.

Theorem

$\mathcal{I}$ is Medvedev equivalent to DomFcn, and hence to $\mathcal{U}$.

Boolean algebras other than the computable sets

There is a Δ^0_2-ultrafilter base for the Boolean algebra of the K-trivial sets.

Modifying the argument above, such an ultrafilter base is necessarily high.

To prove the theorem, we recall the fact by Kučera and Slaman (2009) that there is a Δ^0_2-function h that dominates all functions that are partial computable in some K-trivial set. We use this to modify the construction in proof above that $\mathrm{DomFcn} \geq_s \mathcal{U}$.

Modifying a proof of Jockusch and Stephan (nonhigh cohesive set, 1993) yields:

An oracle C computes an ultrafilter base for the primitive recursive sets iff C' is of PA degree relative to $\emptyset'$.

References:

- Paper with Lempp, Miller and Soskova as CDMTCS research report 547, Auckland University, September 2020, available at https://www.cs.auckland.ac.nz/research/groups/CDMTCS/researchreports/download.php?selected-id=769
- Logic Blog 2019, Section 7; These slides are available

For relational cardinal characteristics in computability see:

- Rupprecht thesis and paper (2010)
- Brendlle, Brooke, Ng, Nies, Proc. Asian Logic Colloq 2014: paper on Cichon diagram in computability
- Greenberg, Kuyper, Turetsky, Comput. 8(3-4): 305-346 (2019): Weihrauch reductions
- Monin and N., Muchnik degrees and cardinal characteristics, JSL, to appear