Department of Mathematics, University of Udine

Cantor-Bendixson theorem in the Weihrauch lattice

Vittorio Cipriani
joint work with Alberto Marcone and Manlio Valenti

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In this talk using the framework of Weihrauch reducibility, we will consider the following classical theorem.

Theorem (Cantor-Bendixson Theorem)

Every closed subset $\mathcal{X}$ of a Polish space can be uniquely written as the disjoint union of a perfect set and a countable set.

The largest perfect subset of $\mathcal{X}$ is called the *perfect kernel* of $\mathcal{X}$ (denoted by $\mathsf{PK}(\mathcal{X})$).
$\mathcal{X} \setminus \mathsf{PK}(\mathcal{X})$ is called the *scattered part* of $\mathcal{X}$.

Theorems as problems

Theorems as the one above can be written as:

$$(\forall x \in X)(\exists y \in Y)(\varphi(x) \to \psi(x, y))$$

and can be naturally translated as a computational problem, i.e.

given in **input** x s.t. $\varphi(x)$, produce as **output** a y s.t. $\psi(x, y)$

N.B. we will show that, for many theorems, there may be many "natural" ways to phrase them as a computational problem.

Represented space

To study computability on some space X we transfer notions of computability in $\mathbb{N}^{\mathbb{N}}$ into X. To do so, we encode each element of X with some $p \in \mathbb{N}^{\mathbb{N}}$.

Definition

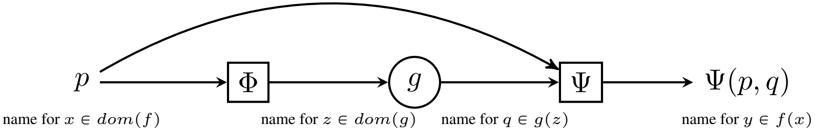
A represented space is a pair (X, δ_X) where $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X$.

$p \in \mathbb{N}^{\mathbb{N}}$ is said to be a *name* for $x \in X$.
Now we can think of a computational problem as a (possibly partial) *multivalued functions* $f :\subseteq X \rightrightarrows Y$, where X, Y are represented spaces.

Weihrauch Reducibility

Let f, g be (partial multivalued) functions on represented spaces.
f is Weihrauch reducible to g ($f \leqslant_W g$) if there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

- Given a name p for $x \in dom(f)$, $\Phi(p)$ is a name for $z \in dom(g)$;
- Given a name q for $w \in g(z)$, $\Psi(p, q)$ is a name for $y \in f(x)$;



In rev. math. we have the so called "big-five phenomenon", that, informally, says that (most of) theorems in classical mathematics are equivalent to one of these five subsystems of SOA:

- $\mathsf{RCA}_0 \rightsquigarrow \mathsf{id}_{\mathbb{N}^\mathbb{N}}$
- $\mathsf{WKL}_0 \rightsquigarrow \mathsf{C}_{2^\mathbb{N}}$
- $\mathsf{ACA}_0 \rightsquigarrow$ (iterations of) lim
- ATR_0
- $\Pi^1_1\text{-}\mathsf{CA}_0$

Which are the representatives (in the Weihrauch lattice) of the big five? Most of the work so far is about the first three.

- $\mathsf{RCA}_0 \rightsquigarrow \mathsf{id}_{\mathbb{N}^\mathbb{N}}$
- $\mathsf{WKL}_0 \rightsquigarrow \mathsf{C}_{2^\mathbb{N}}$
- $\mathsf{ACA}_0 \rightsquigarrow$ (iterations of) lim
- $\mathsf{ATR}_0 \rightsquigarrow \mathsf{C}_{\mathbb{N}^\mathbb{N}}, \mathsf{UC}_{\mathbb{N}^\mathbb{N}}, \dots$
- Π^1_1-$\mathsf{CA}_0 \rightsquigarrow \widehat{\chi_{\Pi^1_1}}$

For Π^1_1-CA_0 the situation is quite clear. We have a natural candidate that is $\widehat{\chi_{\Pi^1_1}}$, where $\chi_{\Pi^1_1}$ is the characteristic function of a Π^1_1-complete set. For ATR_0? Many candidates of different strength.

$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: **Input** an ill-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$
Output a path through T.

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is the restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to trees with a unique path.

Let $\mathcal{X}$ be a computable complete metric space.
For the *scattered part*...

$\mathsf{wScList}_{\mathcal{X}}$: **Input** $A \in \Pi^0_1(\mathcal{X})$.
Output a list $(b_i p_i)_{i\in\omega}$ s.t. $A \setminus \mathsf{PK}(A) = \{p_i : b_i = 1\}$

ScList_{X}: **Input** $A \in \Pi^0_1(\mathcal{X})$.
Output a list $(p_i)_{i\in\omega}$ s.t. $A \setminus \mathsf{PK}(A) = (p_i)_{i\in\omega}$ and $n = |A \setminus \mathsf{PK}(A)|$.

Lemma (Hirst, [1])

$\widehat{\chi_{\Pi^1_1}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathsf{Tr}}$.

Some properties

- $\mathsf{PK}_{\mathsf{Tr}}$ is equivalent both with trees on $2^{<\mathbb{N}}$ and $\mathbb{N}^{<\mathbb{N}}$ (same for $\mathsf{PK}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$);
- $\mathsf{PK}_{\mathsf{Tr}} \equiv_W \widehat{\mathsf{PK}_{\mathsf{Tr}}}$, $\mathsf{PK}_{2^\mathbb{N}} \equiv_W \widehat{\mathsf{PK}_{2^\mathbb{N}}}$ (essentially, given countably many trees, "join" all of them to a new root);

Let $\mathbb{S} = \{\top, \bot\}$ be the Sierpinski space represented via $\delta_\mathbb{S} : \mathbb{N}^\mathbb{N} \to \{\top, \bot\}$ where $\delta_\mathbb{S}^{-1}(\{\bot\}) = \{0^\omega\}$.
Then $\mathbb{S}_{\Pi^1_1} = \{\bot, \top\}$ is the space where $p \in \mathbb{N}^\mathbb{N}$ is a name for $\top$ iff p codes a Π^1_1-complete.
The map $\mathsf{id}(\Pi^1_1, \Sigma^0_1) : \mathbb{S}_{\Pi^1_1} \to \mathbb{S}$ lets us treat a Π^1_1 set as a Σ^0_1 one.

Theorem (C., Marcone, Valenti)

$\mathsf{PK}_{2^\mathbb{N}} \equiv_W \widehat{\mathsf{id}(\Pi^1_1, \Sigma^0_1)}$.

Notice that for $A \in \Pi^0_1(2^{\mathbb{N}})$ we can represent A with a tree $T \subseteq 2^{<\mathbb{N}}$ such that $A = [T]$.
Recall that the collection of trees T s.t. $|[T]| \leqslant \aleph_0$ is Π^1_1-complete.
As $\mathsf{PK}_{2^{\mathbb{N}}}$ is parallelizable, it sufficient to show $\mathsf{id}(\Pi^1_1, \Sigma^0_1) \leqslant_W \mathsf{PK}_{2^{\mathbb{N}}}$.
Let $T \subseteq 2^{<\mathbb{N}}$ be the input for $\mathsf{id}(\Pi^1_1, \Sigma^0_1)$ s.t. Then $\mathsf{id}(\Pi^1_1, \Sigma^0_1) = \top$ iff $|[T]| \leqslant \aleph_0$.
$|[T]| \leqslant \aleph_0$ iff $[\mathsf{PK}_{2^{\mathbb{N}}}(T)] = \emptyset$ (i.e. $\mathsf{PK}_{2^{\mathbb{N}}}(T) \in \mathsf{WF}$). Since $T \in 2^{<\mathbb{N}}$, this is a Σ^0_1 statement.

Let T be the input for $\mathsf{PK}_{2^{\mathbb{N}}}$. Let $T_\sigma = \{\tau \in T : \tau \sqsubset \sigma \vee \sigma \sqsubset \tau\}$. Then we (computably):

- start producing a copy S of our input T and
- for every $\sigma_i \in T$ we compute $\mathsf{id}(\Pi^1_1, \Sigma^0_1)(T_{\sigma_i})$ such that $\mathsf{id}(\Pi^1_1, \Sigma^0_1)(T_{\sigma_i}) = \top$ iff $|[T_{\sigma_i}]| \leqslant \aleph_0$.

For any i let p_i be a name for a solution of $\mathsf{id}(\Pi^1_1, \Sigma^0_1)(T_{\sigma_i})$ (recall that the unique name for $\bot$ is 0^ω).
If there is a n s.t. $p_i(n) \neq 0$, stop adding nodes above σ_i in S.
Since we are removing from S all (infinite extensions of) nodes that in T having countably many paths, we have that S is a suitable name for $\mathsf{PK}_{2^{\mathbb{N}}}(T)$.

Arithmetical Weihrauch reducibility

In this reducibility we let the forward and backward functionals Φ and Ψ to be arithmetical instead of computable.

Corollary

$\mathsf{PK}_{2^{\mathbb{N}}} \equiv^h_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}}$.

Let's turn on other computable complete metric spaces $\mathcal{X}$.
Let $A \in \Pi^0_1(\mathcal{X})$. If given a basic opens set B_i of $\mathcal{X}$, we have that $|B_i \cap A| \leqslant \aleph_0$ is a Π^1_1 property, then the previous proof shows that $\mathsf{PK}_{\mathcal{X}} \leqslant_{\mathrm{W}} \mathsf{id}\widehat{(\Pi^1_1, \Sigma^0_1)}$.

Lemma ((Moschovakis, 3E.6))

For every computable complete metric space $\mathcal{X}$ there is a recursive surjection $\rho : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow \mathcal{X}$ and a Π^0_1 set $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that ρ is one-to-one on A and $\rho[A] = \mathcal{X}$

As we know that that for $A \in \Pi^0_1(\mathbb{N}^{\mathbb{N}})$ the set above is Π^1_1-complete, and $\mathsf{PK}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathbb{N}^{\mathbb{N}}}$, we get the following.

Lemma (C., Marcone, Valenti)

For every computable complete metric space $\mathcal{X}$,
$\mathsf{PK}_{\mathcal{X}} \leqslant_{\mathrm{W}} \mathsf{id}(\widehat{\Pi^1_1, \Sigma^0_1})$.

Recall that a computable metric space $\mathcal{X}$ is called *rich*, if there is a computable injective map $\iota : 2^{\mathbb{N}} \to \mathcal{X}$.

Lemma (C., Marcone, Valenti)

Let $\mathcal{X}$ be a complete rich space. Then $\mathsf{PK}_{2^{\mathbb{N}}} \leqslant_{\mathrm{W}} \mathsf{PK}_{\mathcal{X}}$.

As a corollary, we get that, for all "interesting" spaces, $\mathsf{PK}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathcal{X}}$.

$\mathsf{wCB}_{\mathsf{Tr}}$: **Input** a tree T,
Output $\mathsf{PK}_{\mathsf{Tr}}(T)$ and $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}(T)$.

$\mathsf{CB}_{\mathsf{Tr}}$: **Input** a tree T
Output $\mathsf{PK}_{\mathsf{Tr}}(T)$ and $\mathsf{ScList}_{\mathbb{N}^\mathbb{N}}(T)$.

Lemma (C., Marcone, Valenti)

$\mathsf{wScList} <_{\mathrm{W}} \mathsf{ScList}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}}$.

As $\widehat{\chi_{\Pi^1_1}}$ is trivially parallelizable and $\widehat{\chi_{\Pi^1_1}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathsf{Tr}}$.
As a corollary

$$\widehat{\chi_{\Pi^1_1}} \equiv_{\mathrm{W}} \mathsf{wCB}_{\mathsf{Tr}} \equiv_{\mathrm{W}} \mathsf{CB}_{\mathsf{Tr}}$$

The "full" Cantor-Bendixson (closed sets)

Let $\mathcal{X}$ be a computable complete metric space.

$\mathsf{wCB}_{\mathcal{X}}$: **Input** $A \in \Pi^0_1(\mathcal{X})$.
Output $\mathsf{PK}_{\mathcal{X}}(A)$ and $\mathsf{wScList}_{\mathcal{X}}(A)$.

$\mathsf{CB}_{\mathcal{X}}$: **Input** $A \in \Pi^0_1(\mathcal{X})$.
Output $\mathsf{PK}_{\mathcal{X}}(A)$ and $\mathsf{ScList}_{\mathcal{X}}(A)$.

Lemma (C., Marcone, Valenti)

$\mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{PK}_{2^\mathbb{N}}$.

Recall this lemma.

Lemma ((Moschovakis, 3E.6))

For every computable complete metric space $\mathcal{X}$ there is a recursive surjection $\rho : \mathbb{N}^\mathbb{N} \twoheadrightarrow \mathcal{X}$ and a Π^0_1 set $A \subseteq \mathbb{N}^\mathbb{N}$ such that ρ is one-to-one on A and $\rho[A] = \mathcal{X}$

Corollary (C., Marcone, Valenti)

For every computable complete metric space $\mathcal{X}$, $\mathsf{wScList}_\mathcal{X} \leqslant_\mathrm{W} \mathsf{wScList}_{\mathbb{N}^\mathbb{N}}$. *Similarly,* $\mathsf{ScList}_\mathcal{X} \leqslant_\mathrm{W} \mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$.

As $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \equiv_\mathrm{W} \mathsf{PK}_{2^\mathbb{N}} \equiv_\mathrm{W} \mathsf{PK}_\mathcal{X}$ and since $\mathsf{PK}_{2^\mathbb{N}}$ is parallelizable,

Lemma (C., Marcone, Valenti)

For every complete rich space $\mathcal{X}$, $\mathsf{wCB}_\mathcal{X} \equiv_\mathrm{W} \mathsf{PK}_{2^\mathbb{N}}$.

$\mathsf{CB}_{\mathcal{X}}$ with $\mathcal{X}$ being K_σ

Recall that $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}}$ but $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$.

$\mathcal{X}$ is a computable K_σ-space, if $\mathcal{X}$ is a computable metric space, such that there exists a computable sequence $(K_i)_{i \in \mathbb{N}}$ of non-empty computably compact sets with $X = \bigcup_{i=0}^{\infty} K_i$.

Lemma (C., Marcone, Valenti)

Let $\mathcal{X}$ be a complete rich K_σ-space then $\mathsf{CB}_{\mathcal{X}} \equiv_W \mathsf{PK}_{2^\mathbb{N}}$.

Informally, as $\mathsf{PK}_{2^\mathbb{N}} \equiv_W \mathsf{wCB}_{\mathcal{X}}$ the only information that is missing is the cardinality of the scattered part.Anyway, another instance of $\mathsf{PK}_{2^\mathbb{N}}$ more than suffices to compute it. Then from the cardinality and a "weak" list of the scattered part, we can (computably) get a strong one.

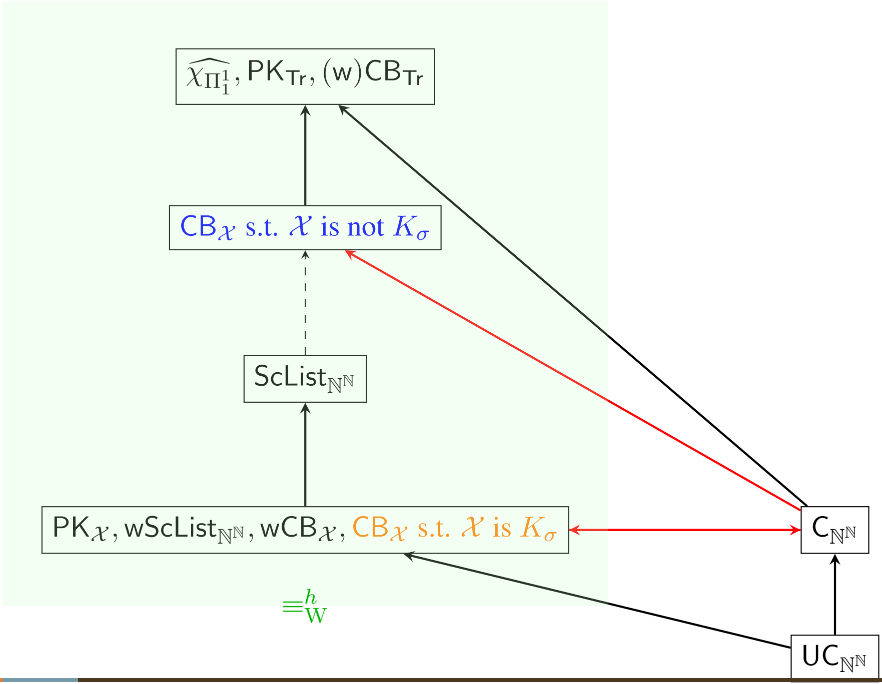
Theorem (Hurewicz, (Kechris 7.10))

Let $\mathcal{X}$ be a Polish space. Then there is an embedding $\iota : \mathbb{N}^\mathbb{N} \to \mathcal{X}$ such that $ran(\mathcal{X})$ is closed iff $\mathcal{X}$ is not K_σ.

The theorem above is not effective. So far, we could get the following:

Lemma (C., Marcone, Valenti)

Let $\mathcal{X}$ be a Polish space that is not K_σ. Then there is an oracle such that $\mathsf{CB}_{\mathcal{X}} \equiv_{\mathrm{W}} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$, relatively to that oracle.


$\widehat{\chi_{\Pi^1_1}}, \mathsf{PK}_{\mathsf{Tr}}, (\mathsf{w})\mathsf{CB}_{\mathsf{Tr}}$
$\mathsf{CB}_{\mathcal{X}}$ s.t. $\mathcal{X}$ is not K_σ
$\mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{PK}_{\mathcal{X}}, \mathsf{wScList}_{\mathbb{N}^\mathbb{N}}, \mathsf{wCB}_{\mathcal{X}}$, $\mathsf{CB}_{\mathcal{X}}$ s.t. $\mathcal{X}$ is K_σ
$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$
$\equiv^h_{\mathrm{W}}$

Separations

Many of the separations come from the characterization of first-order part of a problem f, denoted via ^1f. Introduced by Dzhafarov, Solomon, Yokoyama, the first-order part of a problem is the hardest multi-valued function with codomain $\mathbb{N}$ computed by f.

Soldà and Valenti introduced the $u*$ operator as a generalization of the finite parallelization operator *. Informally given a problem g, g^{u*} solves finitely but unboundedly many instances of g.

T.f.a.e.:

- $^1\mathsf{PK}_{2^{\mathbb{N}}}$, $^1\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$;
- $\Pi^1_1\text{-}\mathsf{C}_{\mathbb{N}}$ where $dom(\Pi^1_1\text{-}\mathsf{C}_{\mathbb{N}}) = \{(T_i)_{i\in\omega} : (\exists i)(T_i \in \mathsf{WF})\}$ and a solution for it is an index n s.t. $T_n \in \mathsf{WF}$.
- $\Pi^1_1\text{-}\mathsf{UC}_{\mathbb{N}}, \Sigma^1_1\text{-}\mathsf{UC}_{\mathbb{N}}, \Delta^1_1\text{-}\mathsf{UC}_{\mathbb{N}}$;
- for all f above, f^{u*};
- $\mathsf{id}(\Pi^1_1, \Sigma^0_1)^{u*}$.

Lemma (C., Marcone, Valenti)

$^1\mathsf{PK}_{\mathcal{X}} \equiv_W \Pi^1_1\text{-}\mathsf{C}_{\mathbb{N}} <_W {}^1\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_W \Sigma^1_1 - \mathsf{C}_{\mathbb{N}} <_W {}^1\widehat{\chi_{\Pi^1_1}} \equiv_W \chi^{*u}_{\Pi^1_1}$

To prove it, just parallelize them and get:

$\widehat{\Pi^1_1\text{-}\mathsf{C}_{\mathbb{N}}} \equiv_W \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_W \widehat{\Sigma^1_1 - \mathsf{C}_{\mathbb{N}}} <_W \widehat{\chi_{\Pi^1_1}}$.

Thanks for your attention!