Comparing induction and bounding principles over RCA_0 and RCA_0^*

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Background on Π^1_2-problems

In reverse mathematics, we often look at Π^1_2-problems.

Definition

A Π^1_2-problem is a sentence

$$\forall X[\Theta(X) \to \exists Y \Psi(X, Y)]$$

of second-order arithmetic such that Θ and Ψ are arithmetic.

Definition

We say that an $X \subseteq \omega$ such that $\Theta(X)$ holds is an instance of this problem, and a solution is a $Y \subseteq \omega$ such that $\Psi(X, Y)$ holds.

We denote such problems by P and Q.

Background on reverse math

We work in second-order arithmetic.

The usual base theory RCA_0 corresponds roughly to computable mathematics.

Formally, RCA_0 consists of the first order axioms for a discrete ordered commutative semiring together with Δ^0_1-comprehension and Σ^0_1-induction.

We will also have occasion to consider RCA_0^*, which is roughly RCA_0 where Σ^0_1 induction is weakened to Σ^0_0 induction.

Reducibility

Definition

We say that P is computably reducible to Q and write P $\leq_c$ Q if for every P-instance X, there is an X -computable instance $\hat{X}$ of Q such that, whenever $\hat{Y}$ is a solution to $\hat{X}$, X has an $X \oplus \hat{Y}$-computable solution.

Definition

We say that P is Weihrauch reducible to Q, P $\leq_W$ Q, if there are Turing functionals Φ and Ψ such that, for every instance X of P, the set Φ^X is an instance of Q, and for every solution $\hat{Y}$ to $\hat{X} = \Phi^X$, the set $Y = \Psi^{X \oplus \hat{Y}}$ is a solution to X.

Using multiple instances

These two reducibilities allow us to use one instance of Q to solve an instance of P. We also sometimes consider ω-reducibility, which allows multiple uses of Q, but in a way that is not uniform. What if we would like to use multiple Q-instances in a uniform way?

Hirschfeldt and Jockusch introduced the idea of a reduction game to allow for this possibility, particularly two-player reduction games for principles P and Q, written as G(Q $\to$ P).

Definition

We say that P is generalized Weihrauch reducible to Q and write P $\leq_{\mathrm{gW}}$ Q, if Player 2 has a computable winning strategy for G(Q $\to$ P).

Extending generalized Weihrauch reducibility

We can extend the notion of a Π^1_2-problem and the game $\mathrm{G}(\mathrm{Q} \to \mathrm{P})$ to a more general setting.
Typically, we extend to the game over RCA_0, $\mathrm{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$.

Definition

We say that P is generalized Weihrauch reducible to Q over RCA_0 and write $\mathrm{P} \leq^{\mathrm{RCA}_0}_{\mathrm{gW}} \mathrm{Q}$, if Player 2 has a computable (i.e., Δ^0_1), winning strategy for $\mathrm{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$.

We define computable reducibility over RCA_0 and Weihrauch reducibility over RCA_0 in a similar way.

The reduction game $\mathrm{G}^{\mathrm{RCA}_0}(\mathrm{Q} \to \mathrm{P})$ captures provability over RCA_0.

Compactness result

Theorem (Dzhafarov, Hirschfeldt, and Reitzes)

Let Γ be a consistent extension of Δ^0_1-comprehension by Π^1_1-formulas that proves the existence of a universal Σ^0_1 formula. Let P and Q be Π^1_2-problems. If $\Gamma \vdash Q \to P$, then there is an n such that Player 2 has a winning strategy for $\hat{\mathrm{G}}^{\Gamma}(Q \to P)$ that ensures victory in at most n many moves. Otherwise, Player 1 has a winning strategy for $\hat{\mathrm{G}}^{\Gamma+Q}(Q \to P)$.

Some relevant bounding principles

In reverse math, we often consider the Σ^0_2-bounding principle $\mathrm{B}\Sigma^0_2$. Over RCA_0, $\mathrm{B}\Sigma^0_2$ is equivalent to the principle Bound^* defined as follows.

Definition

Bound^* is the principle that for a simultaneous enumeration of bounded sets $F_0, \ldots, F_n$, there exists a common bound for the sets F_i.

We will also consider the principle $\mathrm{stBound}^*$, which is a version of Bound^* where the number of sets is not part of the instance.

Induction and bounding reductions cont.

Definition

Let $\mathsf{F}\Sigma^0_1$ denote the Π^1_2 principle: for every Σ^0_1 set A with nonempty complement, there exists an $a \in \overline{A}$ such that either $a = 0$ or $a = S(b)$ for some $b \in A$, where S denotes the successor function.

This is a natural way to think of Σ^0_1 induction as a Π^1_2 principle.

Definition

$\mathsf{C}_{\mathbb{N}}$ is the Π^1_2 principle where an instance is an enumeration of the complement of a nonempty set X, and a solution is an element of X.

Induction and bounding reductions cont.

It is easy to see that $\mathsf{F}\Sigma^0_1 \equiv^{\mathsf{RCA}_0}_{\mathsf{W}} \mathsf{C}\Delta^0_2 \equiv^{\mathsf{RCA}_0}_{\mathsf{W}} \mathsf{C}_{\mathbb{N}}$.
We have shown that

- $\mathsf{F}\Sigma^0_1 \leq^{\mathsf{RCA}^*_0}_{\mathsf{gW}} \mathsf{Bound}^*$,
- $\mathsf{F}\Sigma^0_1 \leq^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{stBound}^*$,
- $\mathsf{C}_{\mathbb{N}} <^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{Bound}^*$,
- $\mathsf{F}\Sigma^0_1 \not\leq^{\mathsf{RCA}^*_0}_{\mathsf{gW}} \mathsf{C}_{\mathbb{N}}$,
- $\mathsf{F}\Sigma^0_1 \not\leq^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{Bound}^*$,
- $\mathsf{Bound}^* \not\leq^{\mathsf{RCA}_0}_{\mathsf{W}} \mathsf{F}\Sigma^0_1$,
- $\mathsf{stBound}^* \not\leq^{\mathsf{RCA}_0}_{\mathsf{W}} \mathsf{F}\Sigma^0_1$,
- $\mathsf{F}\Pi^0_1 \leq^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{F}\Sigma^0_1$, and
- $\mathsf{F}\Delta^0_2 \leq^{\mathsf{RCA}^*_0}_{\mathsf{gW}} \mathsf{Bound}^*$.

Induction and bounding nonreductions

- Every $F\Sigma^0_1$-instance has a solution in RCA_0, but there exist $F\Sigma^0_1$-instances without solutions in RCA_0^*
- $F\Pi^0_1$ does not always have solutions in RCA_0^*
- $C_{\mathbb{N}}$ always has solutions in both RCA_0 and RCA_0^*
- Bound^* and $\mathsf{stBound}^*$ have instances without solutions in both RCA_0 and RCA_0^*
- This gives us many nonreductions, including:
  - $\mathsf{Bound}^* \not\leq^{RCA_0}_{gW} F\Sigma^0_1$
  - $\mathsf{Bound}^* \not\leq^{RCA_0}_{gW} C_{\mathbb{N}}$
  - $F\Sigma^0_1 \not\leq^{RCA_0^*}_{gW} C_{\mathbb{N}}$
  - $F\Pi^0_1 \not\leq^{RCA_0^*}_{gW} C_{\mathbb{N}}$.

Metatheorem

Conditions:

1. Π^1_2 principles P and Q
2. P and Q are first-order (codomain is $\mathbb{N}$)
3. Q has computable instances
4. There exists a computable procedure for computing a number k from X for any P-instance X such that X has a solution between 0 and k
5. For each σ that is an initial segment of a Q-instance and for any finite k and any $n_0, n_1, \ldots, n_k \in \omega$, there is a Q-instance Y extending σ for which $n_0, n_1, \ldots, n_k$ are not solutions

Conclusion:
Q $\not\leq^n_{\mathrm{gW}}$ P for any fixed $n \in \omega$

Metatheorem cont.

Consequently:

- $Q \not\leq_W P$
- $Q \not\leq_{gW}^{RCA_0} P$
- $Q \not\leq_{gW}^{RCA_0^*} P$

Metatheorem proof sketch

- Assume for a contradiction that Q $\leq^n_{\mathrm{gW}}$ P
- Build tree of possible plays of the game based on initial segments σ of Q-instances from S that give convergence to possible initial segments of P-instances for Player 2 to play
- Get value for corresponding k-parameter
- Repeat for each of the first n moves of the game
- On $(n+1)^{st}$ move, get finitely many potential solutions to the Q-instance we're building
- Diagonalize against them using hypothesis on strings in S

Applications of metatheorem

- $X := \{\mathsf{F}\Sigma^0_1, \mathsf{medR}\Sigma^0_1, \mathsf{stR}\Sigma^0_1, \mathsf{stB}\Sigma^0_1\mathsf{CA}, \mathsf{C}_{\mathbb{N}}, \mathsf{C}^*_{\mathbb{N}}, \mathsf{stRT}^1_{<\infty}, \mathsf{Bound}, \mathsf{Bound}^*, \mathsf{stBound}^*, \mathsf{C}\Delta^0_2, \mathsf{F}\Delta^0_2\}$
- $Y := \{\mathsf{R}\Sigma^0_1, \mathsf{B}\Sigma^0_1\mathsf{CA}, \mathsf{K}_{\mathbb{N}}, \mathsf{F}\Pi^0_1, \mathsf{RT}^1_{<\infty}\}$
- For any Q $\in X$ and any P $\in Y$, Q $\not\leq^n_{\mathrm{gW}}$ P for any fixed $n \in \omega$, Q $\not\leq^{\mathsf{RCA}_0}_{\mathrm{gW}}$ P, and Q $\not\leq^{\mathsf{RCA}^*_0}_{\mathrm{gW}}$ P.

In particular

- $\mathsf{F}\Sigma^0_1 \not\leq^n_{\mathrm{gW}} \mathsf{F}\Pi^0_1$
- $\mathsf{C}_{\mathbb{N}} \not\leq^n_{\mathrm{gW}} \mathsf{F}\Pi^0_1$
- $\mathsf{F}\Sigma^0_1 \not\leq^n_{\mathrm{gW}} \mathsf{K}_{\mathbb{N}}$
- $\mathsf{F}\Delta^0_2 \not\leq^n_{\mathrm{gW}} \mathsf{F}\Pi^0_1$

References

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