Effective convergence notions for measures on the real line

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Part I: Background

Weak and Vague Convergence of Measures

$\mathcal{M}(\mathbb{R})$: the space of finite Borel measures on $\mathbb{R}$

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathbb{R})$.

$$\lim_{n\to\infty}\int_{\mathbb{R}} f\,d\mu_n = \int_{\mathbb{R}} f\,d\mu$$

Weak Convergence: $f:\mathbb{R}\to\mathbb{R}$ is a bounded continuous function

Vague Convergence: $f:\mathbb{R}\to\mathbb{R}$ is a continuous function with compact support

Computable Analysis

- A *computable metric space* is a triple (X, d, S) with the following properties:
  - (X, d) is a complete separable metric space
  - $S = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X
  - $d(s_i, s_j)$ is computable uniformly in i, j
- Examples:
  - $(\mathbb{R}, |\cdot|, \mathbb{Q})$
  - $(2^\omega, d_C, S_C)$ where $d_C(X, Y) = 2^{-\min\{n : X(n) \neq Y(n)\}}$ and $S_C = \{\sigma 0^\omega : \sigma \in 2^{<\omega}\}$
- Throughout the talk, we will focus on $X = \mathbb{R}$.

Computable Analysis

- A *(Cauchy) name* of $x \in \mathbb{R}$ is a computable sequence of rationals $\{q_n\}_{n\in\mathbb{N}}$ so that $|q_n - q_{n+1}| < 2^{-n}$.
- A function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable* if there is a Turing functional that sends a name of $x \in \operatorname{dom} f$ to a name of $f(x)$.
- A *(compact-open) name* of a function $f \in C(\mathbb{R})$ is an enumeration ρ_f of the set $\{N_{I,J} : f \in N_{I,J}\}$, where
  - $I \subseteq \mathbb{R}$ is a compact interval;
  - $J \subseteq \mathbb{R}$ is an open interval;
  - $N_{I,J} = \{f \in C(\mathbb{R}) : f(I) \subseteq J\}$.
- A function $F :\subseteq C(\mathbb{R}) \to \mathbb{R}$ is *computable* if there is a Turing functional that sends a name of $f \in \operatorname{dom} F$ to a name of $F(f)$.

$\{I_i\}_{i \in \mathbb{N}}$: effective enumeration of rational open intervals of $\mathbb{R}$

- An open $U \subseteq \mathbb{R}$ is Σ^0_1 if $\{i \in \mathbb{N} : I_i \subseteq U\}$ is c.e.; denote by $\Sigma^0_1(\mathbb{R})$
- A closed $C \subseteq \mathbb{R}$ is Π^0_1 if $\{i \in \mathbb{N} : I_i \cap C = \emptyset\}$ is c.e.; denote by $\Pi^0_1(\mathbb{R})$

$\{W_e\}_{e \in \mathbb{N}}$: effective enumeration of c.e. sets

- Index e of $U \in \Sigma^0_1(\mathbb{R})$: $W_e = \{i \in \mathbb{N} : I_i \subseteq U\}$
- Index e of $C \in \Pi^0_1(\mathbb{R})$: $W_e = \{i \in \mathbb{N} : I_i \cap C = \emptyset\}$

For a compact subset K of $\mathbb{R}$:

- A *(minimal cover)* name of K is an enumeration of all minimal finite open covers of K.
- K is *computably compact* if it has a computable name.
- An index of K is an index of a name of K.

Computable Measure Theory

- A measure $\mu \in \mathcal{M}(\mathbb{R})$ is *computable* if $\mu(\mathbb{R})$ is computable and $\mu(U)$ is left-c.e. uniformly from (an index of) $U \in \Sigma^0_1(\mathbb{R})$.
- A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R})$ is *uniformly computable* if μ_n is a computable measure uniformly in n.
- A set $A \subseteq \mathbb{R}$ is *μ-almost decidable* if there is a pair $U, V \in \Sigma^0_1(\mathbb{R})$ (called a *μ-almost decidable pair*) such that $U \cap V = \emptyset$, $\mu(U \cup V) = \mu(\mathbb{R})$, $\overline{U \cup V} = \mathbb{R}$, and $U \subseteq A \subseteq \mathbb{R} \setminus V$.
- Define an index e of a μ-almost decidable $A \subseteq \mathbb{R}$ to be an index of its μ-almost decidable pair.

Computable Measure Theory

Prokhorov metric ρ on $\mathcal{M}(\mathbb{R})$: $\rho(\mu,\nu) :=$ the infimum over all $\epsilon > 0$ so that $\mu(A) \leq \nu(B(A,\epsilon)) + \epsilon$ and $\nu(A) \leq \mu(B(A,\epsilon)) + \epsilon$ for all $A \in \mathcal{B}(\mathbb{R})$, where

- $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$;
- $B(A,\epsilon) = \bigcup_{a\in A} B(a,\epsilon)$;
- $B(a,\epsilon)$ is the open ball of radius ϵ around a.

Work by M. Hoyrup and C. Rojas (2009) gives us the following:

- $(\mathcal{M}(\mathbb{R}), \rho, \mathcal{D})$ is a computable metric space, where $\mathcal{D}$ denotes the space of finite rational linear combinations of Dirac measures on $\mathbb{R}$.
- $\mu \in \mathcal{M}(\mathbb{R})$ is computable if and only if $I_\mu : f \mapsto \int_{\mathbb{R}} f d\mu$ is computable on computable $f \in C(\mathbb{R})$, uniformly from (a name of) f.

Part II: Effective Weak Convergence of Measures on $\mathbb{R}$

Definitions

- Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathbb{R})$.

Definition.

$\{\mu_n\}_{n\in\mathbb{N}}$ *effectively weakly converges* to $\mu \in \mathcal{M}(\mathbb{R})$ if for every bounded computable function $f :\subseteq \mathbb{R} \to \mathbb{R}$, $\lim_n \int_{\mathbb{R}} f\, d\mu_n = \int_{\mathbb{R}} f\, d\mu$ and it is possible to compute an index of a modulus of convergence for $\{\int_{\mathbb{R}} f\, d\mu_n\}_{n\in\mathbb{N}}$ from an index of f and a bound $B \in \mathbb{N}$ on $|f|$.

Definitions

Definition.

$\{\mu_n\}_{n\in\mathbb{N}}$ *uniformly effectively weakly converges* to $\mu \in \mathcal{M}(\mathbb{R})$ if it weakly converges to μ and there is a uniform procedure that computes for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ a modulus of convergence for $\{\int_{\mathbb{R}} f\, d\mu_n\}_{n\in\mathbb{N}}$ from a name of f and a bound $B \in \mathbb{N}$ on $|f|$.

Example (1)

Fix $a, b \in \mathbb{Q}$, $E \in \mathcal{B}(\mathbb{R})$, a uniformly computable sequence $\{q_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ that decreases to 0. The sequence $\mu_n(E) = \lambda(E \cap [a - q_n, b + q_n])$ effectively weakly converges to $\mu(E) = \lambda(E \cap [a, b])$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

Example (2)

For a uniformly computable sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ that converges to some computable $r \in \mathbb{R}$, the sequence of Dirac measures $\{\delta_{r_n}\}_{n\in\mathbb{N}}$ effectively weakly converges to δ_r.

Properties

Proposition. (McNicholl, R. 2021+)

If $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable and effectively weakly converges to μ, then μ is a computable measure.

Nonexample

Let $\{q_n\}_{n\in\mathbb{N}}$ be a uniformly computable increasing sequence in $\mathbb{Q}$ that converges to an incomputable left-c.e. $\alpha \in \mathbb{R}$. For $E \in \mathcal{B}(\mathbb{R})$ and λ Lebesgue measure on $\mathcal{B}(\mathbb{R})$, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ defined by $\mu_n(E) = \lambda(E \cap [0, q_n])$ weakly converges to $\mu(E) = \lambda(E \cap [0, \alpha])$, but fails to effectively weakly converge since $\mu(\mathbb{R}) = \lambda([0, \alpha]) = \alpha$ is not computable.

Properties

Theorem. (McNicholl, R. 2021+)

Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. The following are equivalent:

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively weakly convergent;

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly effectively weakly convergent.

Effective Portmanteau Theorem

Portmanteau Theorem (Alexandroff 1941)

For a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R})$, the following are equivalent.

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ weakly converges to μ

(2) For every uniformly continuous $f \in C_b(\mathbb{R})$, $\lim_{n\to\infty} \int_{\mathbb{R}} f d\mu_n = \int_{\mathbb{R}} f d\mu$.

(3) For every closed $C \subseteq \mathbb{R}$, $\limsup_{n\to\infty} \mu_n(C) \leq \mu(C)$.

(4) For every open $U \subseteq \mathbb{R}$, $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$.

(5) For every μ-continuity $A \subseteq \mathbb{R}$, $\lim_{n\to\infty} \mu_n(A) = \mu(A)$.

Effective Portmanteau Theorem

To help us formulate an effective version of the aforementioned theorem, we need the following definition.

Definition.

Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of reals, and let $g :\subseteq \mathbb{Q} \to \mathbb{N}$.

1. We say g *witnesses that* $\liminf_n a_n$ *is not smaller than* a if $\mathrm{dom}(g)$ is the left Dedekind cut of a and if $r < a_n$ whenever $r \in \mathrm{dom}(g)$ and $n \geq g(r)$.
2. We say g *witnesses that* $\limsup_n a_n$ *is not larger than* a if $\mathrm{dom}(g)$ is the right Dedekind cut of a and if $r > a_n$ whenever $r \in \mathrm{dom}(g)$ and $n \geq g(r)$.

Effective Portmanteau Theorem

Theorem. (McNicholl, R. 2021+)

For a uniformly computable sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R})$, the following are equivalent.

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ effectively weakly converges to μ

(2) From $e, B \in \mathbb{N}$ so that e indexes a uniformly continuous $f \in C_b(\mathbb{R})$ with $|f| \leq B$, it is possible to compute a modulus of convergence of $\{\int_{\mathbb{R}} f\, d\mu_n\}_{n\in\mathbb{N}}$ with limit $\int_{\mathbb{R}} f\, d\mu$.

(3) μ is computable, and from an index of $C \in \Pi^0_1(\mathbb{R})$ it is possible to compute an index of a witness that $\limsup_n \mu_n(C)$ is not larger than $\mu(C)$.

(4) μ is computable, and from an index of $U \in \Sigma^0_1(\mathbb{R})$ it is possible to compute an index of a witness that $\liminf_n \mu_n(U)$ is not smaller than $\mu(U)$.

(5) μ is computable, and for every μ-almost decidable A, $\lim_n \mu_n(A) = \mu(A)$ and an index of a modulus of convergence of $\{\mu_n(A)\}_{n\in\mathbb{N}}$ can be computed from an index of A.

Effective Convergence in the Prokhorov Metric

- We say $\{\mu_n\}_{n\in\mathbb{N}}$ *converges effectively in the Prokhorov metric* ρ to μ if there is a computable function $\epsilon : \mathbb{N} \to \mathbb{N}$ such that $n \geq \epsilon(N)$ implies $\rho(\mu_n, \mu) < 2^{-N}$ for all n, N.
- Note: ρ metrizes the topology of weak convergence of measures on $\mathcal{M}(X)$ for a separable metric space X.
- The following result is a consequence of the Effective Portmanteau Theorem.

Theorem. (R. 2021+)

For a uniformly computable sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R})$, the following are equivalent.

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ effectively weakly converges to μ

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ converges effectively in ρ to μ

Part III: Effective Vague Convergence of Measures on $\mathbb{R}$

Definitions

- Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathbb{R})$.

Definition.

$\{\mu_n\}_{n\in\mathbb{N}}$ *effectively vaguely converges* to $\mu \in \mathcal{M}(\mathbb{R})$ if for every compactly-supported computable function $f :\subseteq \mathbb{R} \to \mathbb{R}$, $\lim_n \int_{\mathbb{R}} f\,d\mu_n = \int_{\mathbb{R}} f\,d\mu$ and it is possible to compute an index of a modulus of convergence for $\{\int_{\mathbb{R}} f\,d\mu_n\}_{n\in\mathbb{N}}$ from an index of f and an index of $\operatorname{supp} f$.

Definitions

Definition.

$\{\mu_n\}_{n\in\mathbb{N}}$ *uniformly effectively vaguely converges* to $\mu \in \mathcal{M}(\mathbb{R})$ if it vaguely converges to μ and there is a uniform procedure that computes for any compactly-supported continuous function $f : \mathbb{R} \to \mathbb{R}$ a modulus of convergence for $\{\int_{\mathbb{R}} f \, d\mu_n\}_{n\in\mathbb{N}}$ from a name of f and a name of $\operatorname{supp} f$.

Properties

In contrast to effective weak convergence:

Proposition. (R. 2021+)

There is a uniformly computable sequence in $\mathcal{M}(\mathbb{R})$ that effectively vaguely converges but such that the limit measure μ has the property that $\mu(\mathbb{R})$ is an incomputable real.

Sketch.
Let $A \subset \mathbb{N}$ be an incomputable c.e. set, and let $\{a_i\}_{n\in\mathbb{N}}$ be an effective enumeration of A. The sequence $\mu_n = \sum_{i=0}^{n} 2^{-(a_i+1)}\delta_i$ for each $n \in \mathbb{N}$ effectively vaguely converges to the measure $\mu = \sum_{i=0}^{\infty} 2^{-(a_i+1)}\delta_i$. Note that $\mu(\mathbb{R}) = \sum_{i=0}^{\infty} 2^{-(a_i+1)}$ is incomputable since it is the limit of a Specker sequence. $\square$

Properties

Proposition. (R. 2021+)

If $\{\mu_n\}_{n\in\mathbb{N}}$ is a uniformly computable sequence that effectively vaguely converges to μ and $\mu(\mathbb{R})$ is computable, then μ is computable.

Theorem. (R. 2021+)

Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. The following are equivalent:

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively vaguely convergent;

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly effectively vaguely convergent.

Connections with Effective Weak Convergence

Theorem. (R. 2021+)

Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. Suppose further that there is a computable modulus of convergence for $\{\mu_n(\mathbb{R})\}_{n\in\mathbb{N}}$. The following are equivalent:

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively vaguely convergent;

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively weakly convergent.

Corollary.

Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is a uniformly computable sequence of probability measures. The following are equivalent:

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively vaguely convergent;

(2) $\{\mu_n\}_{n\in\mathbb{N}}$ is effectively weakly convergent.

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Thank you!