Relative topos theory via stacks

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Relativity techniques

- Broadly speaking, in Mathematics the relativization method consists in trying to state notions and results in terms of morphisms, rather than objects, of a given category, so that they can be 'relativized' to an arbitrary base object.

- One works in the new, relative universe as it were the 'classical' one, and then interprets the obtained results from the point of view of the original universe. This process is usually called *externalization*.

- Relativity techniques can be thought as general 'change of base techniques', allowing one to choose the universe relatively to which one works according to one's needs.

- The relativity method has been pionneered by Grothendieck, in particular for schemes, in his categorical refoundation of Algebraic Geometry, and has played a key role in his work.

- We aim for a similar set of tools for toposes, that is, for an efficient formalism for doing topos theory over an arbitrary base topos.

Topos theory over an arbitrary base topos

Our new foundations for relative topos theory are based on stacks
(and, more generally, fibrations and indexed categories).

The approach of category theorists (Lawvere, Diaconescu,
Johnstone, etc.) to this subject is chiefly based on the notions of
internal category and of internal site.

The problem with these notions is that they are too rigid to
naturally capture relative topos-theoretic phenomena, as well as
for making computations and formalizing 'parametric reasoning'.

We shall resort to the more general and technically flexible notion
of stack, developing the point of view originally introduced by J.
Giraud in his paper *Classifying topos*.

Grothendieck topologies

Recall that a sieve on an object c of a category $\mathcal{C}$ is a collection of arrows to c which is closed with respect to composition on the right.

Definition

- A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c, called the J-covering sieves, in such a way that
    (i) (maximality axiom) the maximal sieve $M_c = \{f \mid \mathrm{cod}(f) = c\}$ is in $J(c)$;
    (ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;
    (iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

- A site (resp. small site) is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category (resp. a small category) and J is a Grothendieck topology on $\mathcal{C}$.

- A site $(\mathcal{C}, J)$ is said to be small-generated if $\mathcal{C}$ is locally small and has a small J-dense subcategory (that is, a category $\mathcal{D}$ such that every object of $\mathcal{C}$ admits a J-covering sieve generated by arrows whose domains lie in $\mathcal{D}$, and for every arrow $f : d \to c$ in $\mathcal{C}$ where d lies in $\mathcal{D}$ the family of arrows $g : \mathrm{dom}(g) \to d$ such that $f \circ g$ lies in $\mathcal{D}$ generates a J-covering sieve).

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Examples of Grothendieck topologies

- If X is a topological space, the usual notion of covering in Topology gives rise to the following Grothendieck topology $J_{\mathcal{O}(X)}$ on the poset category $\mathcal{O}(X)$: for a sieve $S = \{U_i \hookrightarrow U \mid i \in I\}$ on $U \in Ob(\mathcal{O}(X))$,

$$S \in J_{\mathcal{O}(X)}(U) \text{ if and only if } \bigcup_{i \in I} U_i = U .$$

- If $\mathcal{C}$ satisfies the dual of the amalgamation property then the atomic topology on $\mathcal{C}$ is the topology J_{at} defined by: for a sieve S,

$$S \in J_{at}(c) \text{ if and only if } S \neq \emptyset .$$

- The Zariski topology on the opposite of the category **Rng**$_{f.g.}$ of finitely generated commutative rings with unit is defined by: for any cosieve S in **Rng**$_{f.g.}$ on an object A, $S \in Z(A)$ if and only if S contains a finite family $\{\xi_i : A \to A_{f_i} \mid 1 \leq i \leq n\}$ of canonical maps $\xi_i : A \to A_{f_i}$ in **Rng**$_{f.g.}$ where $\{f_1, \ldots, f_n\}$ is a set of elements of A which is not contained in any proper ideal of A.

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Sheaves on a site

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

- Let $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d .$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S .$$

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a *J-sheaf* if every matching family for any *J*-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.

- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are *J*-sheaves.

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The notion of Grothendieck topos

Definition
A Grothendieck topos is any category equivalent to the category of sheaves on a small(-generated) site.

The following examples show that toposes can be naturally attached to mathematical notions as different as (small) categories, topological spaces, or groups:

Examples

- For any (small) category $\mathcal{C}$, $[\mathcal{C}^{op}, \textbf{Set}]$ is the category of sheaves $\textbf{Sh}(\mathcal{C}, T)$ where T is the trivial topology on $\mathcal{C}$.

- For any topological space X, $\textbf{Sh}(\mathcal{O}(X), J_{\mathcal{O}(X)})$ is equivalent to the usual category $\textbf{Sh}(X)$ of sheaves on X.

- For any (topological) group G, the category $BG = \textbf{Cont}(G)$ of continuous actions of G on discrete sets is a Grothendieck topos (equivalent to the category $\textbf{Sh}(\textbf{Cont}_t(G), J_{at})$ of sheaves on the full subcategory $\textbf{Cont}_t(G)$ on the non-empty transitive actions with respect to the atomic topology).

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Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism:

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

Grothendieck toposes, geometric morphisms and geometric transformations form a 2-category, called **Topos**.

Example

A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors
between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$ is a functor $F : \mathcal{C} \to \mathcal{C}'$
  such that there is a geometric morphism
  $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square
  commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ F\ } & \mathcal{C}' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{\ u^*\ } & \mathbf{Sh}(\mathcal{C}', J'); \end{array}$$

- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$
  which has the covering-lifting property (in the sense that for
  any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a
  K-covering sieve R on d such that $\pi(R) \subseteq S$).

Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a
  geometric morphism* $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a
  geometric morphism* $C_\pi : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

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Continuous functors

Another important class of functors between sites is that of continuous ones:

Definition (Grothendieck)

Given sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be (J, K)-continuous, or simply, continuous, if the functor

$$D_A := (- \circ A^{\mathrm{op}}) : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

The property of continuity of a functor can be interpreted as a form of cofinality; in fact, we have shown that it can be explicitly characterized in terms of relative cofinality conditions.

Recall that every continuous comorphism of sites p induces an essential geometric morphism C_p.

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Indexed categories and fibrations

The language in which we shall work for developing relative topos theory is that of indexed categories and fibrations.

- Given a category $\mathcal{C}$, we shall denote by **Ind**$_\mathcal{C}$ the 2-category of $\mathcal{C}$-indexed categories: it is the 2-category $[\mathcal{C}^{op}, \textbf{Cat}]_{ps}$ whose 0-cells are the pseudofunctors $\mathcal{C}^{op} \to \textbf{Cat}$, whose 1-cells are the pseudonatural transformations and whose 2-cells are the modifications between them.

- Given a category $\mathcal{C}$, we shall denote by **Fib**$_\mathcal{C}$ the 2-category of fibrations over $\mathcal{C}$: it is the sub-2-category of **CAT**$/\mathcal{C}$ whose 0-cells are the (Street) fibrations $p : \mathcal{D} \to \mathcal{C}$, whose 1-cells are the morphisms of fibrations (with a 'commuting' isomorphism) and whose 2-cells are the natural transformations between them.

  We shall denote by **cFib**$_\mathcal{C}$ the full sub-2-category of cloven fibrations (i.e. fibrations equipped with a cleavage).

It is well-known that indexed categories and fibrations are in equivalence with each other:

Theorem
*For any category $\mathcal{C}$, there is an equivalence of 2-categories between **Ind**$_\mathcal{C}$ and **cFib**$_\mathcal{C}$, one half of which is given by the Grothendieck construction and whose other half is given by the functor taking the fibers at the objects of $\mathcal{C}$.*

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The notion of stack

Definition

Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a *J-prestack* (resp. *J-stack*) if for every *J*-sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$- \circ \textstyle\int m_S : \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\textstyle\int S, \mathcal{D})$$

is full and faithful (resp. an equivalence).

Stacks over a site $(\mathcal{C}, J)$ form a 2-full and faithful subcategory of $\mathbf{Ind}_{\mathcal{C}}$, which we will denote by $\mathbf{St}(\mathcal{C}, J)$.

The notion of stack on a site is a higher-categorical generalization of that of sheaf on that site:

Proposition

Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$: then P is J-separated (resp. J-sheaf) if and only if the fibration $\int P \to \mathcal{C}$ is a J-prestack (resp. J-stack).

We can rewrite the condition for a pseudofunctor $\mathcal{C}^{op} \to \mathbf{Cat}$ to be a *J*-prestack (resp. *J*-stack) in the language of indexed categories, as the requirement that for every sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(y_{\mathcal{C}}(X), \mathbb{D}) \xrightarrow{- \circ m_S} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{D})$$

be full and faithful (resp. an equivalence), where both $y_{\mathcal{C}}(X)$ and S are interpreted as discrete $\mathcal{C}$-indexed categories.

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Fibrations as comorphisms of sites

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K in $\mathcal{D}$, there is a smallest Grothendieck topology M_K^A on $\mathcal{C}$ which makes A a comorphism of sites to $(\mathcal{D}, K)$.

Proposition (O.C. and R.Z.)

If A is a fibration, the topology M_K^A admits the following simple description: a sieve R is M_K^A-covering if and only if the collection of cartesian arrows in R is sent by A to a K-covering family.

We shall call M_K^A the Giraud topology induced by K, in honour of Jean Giraud, who used it for constructing the classifying topos $\mathbf{Sh}(\mathcal{C}, M_K^A)$ of a stack A on $(\mathcal{D}, K)$.

Proposition (O.C.)

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{C}', M_K^{A'})$.

In particular, a fibration $A : \mathcal{C} \to \mathcal{D}$ yields a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{D}, K)$ for any Grothendieck topology K on $\mathcal{D}$.

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Unifying morphisms and comorphisms of sites

We can unify the notions of morphism and comorphisms of sites
by interpreting them as two fundamentally different ways of
describing morphisms of toposes which correspond to each other
under a topos-theoretic 'bridge'.

More specifically, morphisms of sites provide an 'algebraic'
perspective on morphisms of toposes, while comorphisms of sites
provide a 'geometric' perspective on them.

The key idea is to replace the given sites of definition with
Morita-equivalent ones in such a way that the given morphism
(resp. comorphism) of sites acquires a left (resp. right) adjoint,
not necessarily in the classical categorial sense but in the weaker
topos-theoretic sense of the associated comma categories having
equivalent associated toposes.

Let us focus on the procedure for turning a morphism of sites into
a comorphism of sites inducing the same geometric morphism.

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From morphisms to comorphisms of sites

Theorem (O.C.)

Given a morphism $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ of small-generated sites, let

- *$(1_{\mathcal{D}} \downarrow F)$ be the 'comma category' whose objects are the triplets $(d, c, \alpha : d \to F(c))$*

- *i_F be the functor $\mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ sending any object c of $\mathcal{C}$ to the triplet $(F(c), c, 1_{F(c)})$,*

- *$\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ and $\pi_{\mathcal{D}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ the canonical projection functors, and*

- *$\tilde{K}$ be the Grothendieck topology on $(1_{\mathcal{D}} \downarrow F)$ whose covering sieves are those whose image under $\pi_{\mathcal{D}}$ is K-covering.*

Then:

(i) *$\pi_{\mathcal{C}} \dashv i_F$, $\pi_{\mathcal{D}} \circ i_F = F$, i_F is a morphism of sites $(\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ and $c_F := \pi_{\mathcal{C}}$ is a comorphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$.*

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(ii) $\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences

$$C_{\pi_{\mathcal{D}}} : \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$
\begin{array}{ccc}
\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) & \underset{\overset{\sim}{\underset{\mathbf{Sh}(\pi_{\mathcal{D}})}{\longleftarrow}}}{\overset{C_{\pi_{\mathcal{D}}}}{\longrightarrow}} & \mathbf{Sh}(\mathcal{D}, K) \\
& & \\
C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_F) \searrow & & \swarrow \mathbf{Sh}(F) \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{array}
$$

For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, f^* is a morphism of sites $(\mathcal{E}, J_{\mathcal{E}}^{\text{can}}) \to (\mathcal{F}, J_{\mathcal{F}}^{\text{can}})$ such that $f = \mathbf{Sh}(f^*)$. We thus obtain the following result.

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The canonical stack of a geometric morphism

Corollary (O.C.)

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

is a comorphism of sites $((1_{\mathcal{F}} \downarrow f^*), \widetilde{J_{\mathcal{F}}^{can}}) \to (\mathcal{E}, J_{\mathcal{E}}^{can})$ such that $f = C_{\pi_{\mathcal{E}}}$.

The functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of f: from an indexed point of view, this stack sends any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.

By taking f to be the identity, and choosing a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$, we obtain the canonical stack $\mathcal{S}_{(\mathcal{C}, J)}$ on $(\mathcal{C}, J)$, which sends any object c of $\mathcal{C}$ to the topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$. The above corollary thus specializes to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, \widetilde{J_{\mathbf{Sh}(\mathcal{C},J)}^{can}}),$$

which represents a 'thickening' of the usual representation of a Grothendieck topos as the topos of sheaves over itself with respect to the canonical topology.

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Stacks over a site

The role of stacks in our approach to relative topos theory is two-fold:

- On the one hand, the notion of stack represents a higher-order categorical generalization of the notion of sheaf. Accordingly, categories of stacks on a site represent higher-categorical analogues of Grothendieck toposes. One can thus expect to be able to lift a number of notions and constructions pertaining to sheaves (resp. Grothendieck toposes) to stacks (resp. categories of stacks on a site).

- On the other hand, stacks on a site $(\mathcal{C}, J)$ generalize internal categories in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Since (usual) categories can be endowed with Grothendieck topologies, so stacks on a site can also be endowed with suitable analogues of Grothendieck topologies. This leads to the notion of *site relative to a base topos*, which is crucial for developing relative topos theory.

Remark
Every stack is equivalent to a split stack, that is to an internal category, but most stacks naturally arising in the mathematical practice are not split (think, for instance, of the canonical site of a topos).

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Our theory is based on a network of 2-adjunctions, as follows:

$$
\begin{array}{ccc}
\mathbf{Ind}_{\mathcal{C}} & \underset{\Gamma}{\overset{\Lambda}{\underset{\perp}{\rightleftarrows}}} & \mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \\
{\scriptstyle \vdash}\big\downarrow s_J & & \big\uparrow \\
\mathbf{St}(\mathcal{C}, J) & \underset{\Gamma'}{\overset{\Lambda'}{\underset{\perp}{\rightleftarrows}}} & \mathbf{EssTopos}/\mathbf{Sh}(\mathcal{C}, J)^{co} \\
{\scriptstyle \vdash}\big\downarrow E\circ\Lambda' & \overset{E}{\underset{L}{\rightleftarrows}} & \\
\mathbf{Sh}(\mathcal{C}, J) & &
\end{array}
$$

In this diagram, $(\mathcal{C}, J)$ is a small-generated site, $\mathbf{Ind}_{\mathcal{C}}$ is the category of $\mathcal{C}$-indexed categories, $\mathbf{Sh}(\mathcal{C}, J)$ is the category of J-sheaves on $\mathcal{C}$, $\mathbf{St}(\mathcal{C}, J)$ is the category of J-stacks on $\mathcal{C}$, s_J is the stackification functor, $\mathbf{Topos}$ is the category of Grothendieck toposes and geometric morphisms and $\mathbf{EssTopos}$ is the full subcategory on the essential geometric morphisms.

The functor E sends an essential geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the object $f_!(1_{\mathscr{E}})$ (where $f_!$ is the left adjoint to the inverse image f^* of f) and the functor L sends an object P of $\mathbf{Sh}(\mathcal{C}, J)$ to the canonical local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathbf{Sh}(\mathcal{C}, J)$.

Relative 'presheaf toposes'

Given a $\mathcal{C}$-indexed category $\mathbb{D}$, we denote by $\mathcal{G}(\mathbb{D})$ the fibration on $\mathcal{C}$ associated with it (through the Grothendieck construction) and by $p_{\mathbb{D}}$ the canonical projection functor $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

Proposition (O.C. and R.Z.)

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}$ a $\mathcal{C}$-indexed category and $\mathbb{D}^V$ be the opposite indexed category of $\mathbb{D}$ (defined by setting, for each $c \in \mathcal{C}$, $\mathbb{D}^V(c) = \mathbb{D}(c)^{\mathrm{op}}$). Then we have a natural equivalence

$$\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}) \ .$$

This proposition shows that, if $\mathbb{D}$ is a stack, the classifying topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of $\mathbb{D}$, which we call the Giraud topos of $\mathbb{D}$, can indeed be seen as the "topos of relative presheaves on $\mathbb{D}$".

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Giraud toposes as weighted colimits

We have shown that, for any $\mathbb{D}$, the Giraud topos
$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \to \mathbf{Sh}(\mathcal{C}, J)$ can be naturally seen as a
weighted colimit of a diagram of étale toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

where $y : Y \to X$ and $a : U \to V$ are arrows respectively in $\mathcal{C}$ and
in $\mathbb{D}(X)$, the legs $\lambda_{(X,U)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{G}(\mathbb{D}, M_J^{p_{\mathbb{D}}})$ of the
cocone are the morphisms $C_{\xi_{(X,U)}}$ induced by the morphisms of
fibrations $\xi_{(X,U)} : \mathcal{C}/X \to \mathcal{D}$ over $\mathcal{C}$ given by the fibered Yoneda
lemma, and the functor $\Sigma_y : \mathcal{C}/Y \to \mathcal{C}/X$ are given by composition
with y.

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The fundamental adjunction

The universal property of the above weighted colimit yields a
fundamental 2-adjunction between cloven fibrations over $\mathcal{C}$ and
toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

Theorem (O.C and R.Z.)

For any small-generated site $(\mathcal{C}, J)$, the two pseudofunctors

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\left[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\phi)} [q : \mathcal{E} \to \mathcal{C}] \right] \mapsto \left[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\phi)} [\mathrm{Gir}_J(q)] \right],$$

and

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}_{\mathcal{C}} \simeq \mathbf{cFib}_{\mathcal{C}},$$

$$[E : \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)] \mapsto \left[\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{op} \to \mathbf{CAT} \right]$$

are the two components of a 2-adjunction

$$\mathbf{cFib}_{\mathcal{C}} \underset{\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)$$

Remark

*Since $\mathrm{Gir}_J(p) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathcal{D}^V, \mathcal{S}_{(\mathcal{C},J)})$, the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ has a
similar behavior to that of a dualizing object for the adjunction $\Lambda \dashv \Gamma$.*

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The discrete setting

The specialization of our fundamental adjunction in the setting of presheaves (that is, of discrete fibrations) yields a generalization to the context of arbitrary sites of the classical adjunction

$$\mathbf{Psh}(X) \underset{\Gamma}{\overset{\Lambda}{\underset{\perp}{\rightleftarrows}}} \mathbf{Top}/X \ .$$

between presheaves on a topological space X and bundles over it.

Definition

We call a geometric morphism $F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)$ small relative to $\mathbf{Sh}(\mathcal{C}, J)$ if for any J-sheaf $P : \mathcal{C}^{op} \to \mathbf{Set}$ the geometric morphisms $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathscr{F}$ over $\mathbf{Sh}(\mathcal{C}, J)$ form a set (up to equivalence of geometric morphisms), that is, if the category

$$\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/P, \mathscr{F})$$

is small.

We denote by $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ the full subcategory of the 1-category $\mathbf{Topos}/_1\mathbf{Sh}(\mathcal{C}, J)$ whose objects are the small geometric morphisms relative to $\mathbf{Sh}(\mathcal{C}, J)$.

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Proposition (O.C. and R.Z.)

Consider a small-generated site $(\mathcal{C}, J)$:

- *There is an adjunction of 1-categories*

$$[\mathcal{C}^{op}, \mathbf{Set}] \underset{\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)}}{\overset{\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)}}{\rightleftarrows}} \mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J) \quad .$$

- *The functor $\Lambda_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)}$ maps a presheaf P to $\coprod_{a_J(P)} : \mathbf{Sh}(\mathcal{C}, J)/a_J(P) \to \mathbf{Sh}(\mathcal{C}, J)$ or, in terms of comorphisms of sites, to $\Lambda(P) := [C_{p_P} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\mathcal{C}, J)]$ and $\Lambda(g) := C_{\int g} : \mathbf{Sh}(\int P, J_P) \to \mathbf{Sh}(\int Q, J_Q)$.*

- *The functor $\Gamma_{\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)}$ acts like a Hom-functor by mapping an object $[F : \mathscr{F} \to \mathbf{Sh}(\mathcal{C}, J)]$ of $\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)$ to the presheaf*

$$\mathbf{Topos}^s/_1\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{F}) : \mathcal{C}^{op} \to \mathbf{Set} \ .$$

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The general presheaf-étale adjunction

- The image of $\Lambda_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)}$ factors through
  $\textbf{Topos}^{\acute{e}tale}/\textbf{Sh}(\mathcal{C},J)$, and the image of $\Gamma_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)}$ factors
  through $\textbf{Sh}(\mathcal{C},J)$;

- The fixed points of $\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)$ are precisely the étale
  geometric morphisms, while those of $[\mathcal{C}^{op}, \textbf{Set}]$ are
  J-sheaves.

- The adjunction $\Lambda_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)} \dashv \Gamma_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)}$ restricts to an
  equivalence

$$\textbf{Sh}(\mathcal{C},J) \simeq \textbf{Topos}^{\acute{e}tale}/_1\textbf{Sh}(\mathcal{C},J) \ .$$

- The composite functor $\Gamma_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)}\Lambda_{\textbf{Topos}^s/_1\textbf{Sh}(\mathcal{C},J)}$ is
  naturally isomorphic to the sheafification functor

$$i_J a_J : [\mathcal{C}^{op}, \textbf{Set}] \to \textbf{Sh}(\mathcal{C},J) \to [\mathcal{C}^{op}, \textbf{Set}];$$

Some applications

The presheaf-bundle adjunction for topological spaces is useful mostly becase it provides a geometric interpretation of several fundamental constructions on (pre)sheaves, such as direct and inverse images, as well as the sheafification process, in the language of fibrations.

Thanks to our site-theoretic generalization, we can extend these techniques to arbitrary presheaves. In particular, we obtain the following results:

- For any $c \in \mathcal{C}$, the elements $a_J(P)(c)$ of the J-sheafification of a given presheaf P can be identified with the geometric morphisms over $\mathbf{Sh}(\mathcal{C}, J)$ from $\mathbf{Sh}(\mathcal{C}/c, J_c)$ to $\mathbf{Sh}(\int P, J_P)$, all of which can be locally represented as being induced by morphisms of fibrations.

  This is strictly related to the construction of $a_J(P)(c)$ in terms of locally matching families of elements of P.

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Direct and inverse images in terms of fibrations

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ and two presheaves $P : \mathcal{C}^{op} \to$ **Set** and $Q : \mathcal{D}^{op} \to$ **Set** with associated fibrations $\pi_P : \int P \to \mathcal{C}$ and $\pi_Q : \int Q \to \mathcal{D}$,
  - the fibration corresponding to the direct image presheaf $Q \circ F^{op}$ is computed as the strict pullback of π_Q along F:

  $$
  \begin{array}{ccc}
  \int(F^*(Q)) & \longrightarrow & \int Q \\
  \downarrow & \lrcorner & \downarrow \pi_Q \\
  \mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D}
  \end{array}
  $$

  - If F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ then, for any J-sheaf P on $\mathcal{C}$, the inverse image **Sh**$(F)^*(P)$ coincides with the discrete part of the K-comprehensive factorization (in the sense of O.C.) of the composite functor $F \circ \pi_P$.

We have also established natural analogues of these results in the context of stacks.

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Relative sheaf toposes

As any Grothendieck topos is a subtopos of a presheaf topos, so any relative topos should be a subtopos of a relative presheaf topos. This motivates the following

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A site relative to $(\mathcal{C}, J)$ is a pair consisting of a $\mathcal{C}$-indexed category $\mathbb{D}$ and a Grothendieck topology K on $\mathcal{G}(\mathbb{D})$ which contains the Giraud topology $M_J^{p_{\mathbb{D}}}$.

The topos of sheaves on such a relative site $(\mathbb{D}, K)$ is defined to be the geometric morphism

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by the comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Remark

Not every Grothendieck topology on K can be generated starting by horizontal or vertical data (that is, by sieves generated by cartesian arrows or entirely lying in some fiber), but many important relative topologies naturally arising in practice are of this form.

Examples of relative topologies

- The Giraud topology is an example of a relative topology generated by horizontal data.

- The total topology of a fibered site, in the sense of Grothendieck, is generated by vertical data.

- The topology presenting the over-topos at a model (introduced in a joint work with Axel Osmond), defined on the stack of its generalized elements, is an example of a 'mixed' relative topology.

We have shown that, for a wide class of relative topologies generated by horizontal and vertical data, one can describe bases for them consisting of multicompositions of horizontal families with vertical families, thus generalizing the description of bases provided in the context of the over-topos construction.

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A problem of Grothendieck

As recently brought to the public attention by Colin McLarty, Grothendieck expressed, in his 1973 Buffalo lectures, the aspiration of viewing any object of a topos geometrically as an étale space over the terminal object:

> *The intuition is the following: viewing objects of a topos as being something like étale spaces over the final object of the topos, and the induced topos over an object as just the object itself. That is I think the way one should handle the situation.*
>
> *It's a funny situation because in strict terms, you see, the language which I want to push through doesn't make sense. But of course there are a number of mathematical statements which substantiate it.*

Given his conception of *gros* and *petit* toposes, we can more broadly interpret his wish as that for a framework allowing one to think geometrically about any topos, that is, as it were a 'petit' topos related to a 'gros' topos by a local retraction.

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Local morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be local if f_* has a fully faithful right adjoint.

Theorem (O.C.)

Let $F : \mathcal{D} \to \mathcal{C}$ be a bimorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$. Then:

(i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is essential, and*

$$(C_F)_! \cong \mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \cong (C_F)^* = D_F := (- \circ F^{\mathrm{op}}) \dashv (C_F)_*$$

(ii) *The morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is local if and only if C_F is an inclusion, that is, if and only if F is K-faithful and K-full. In this case, the morphisms C_F and $\mathbf{Sh}(F)$ realize the topos $\mathbf{Sh}(\mathcal{D}, K)$ as a (coadjoint) retract of $\mathbf{Sh}(\mathcal{C}, J)$ in $\mathbf{Topos}$.*

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Gros and *petit* toposes

Pairs of *gros* and *petit* toposes are important for several reasons. Morally, a *petit* topos is thought of as a generalized space, while a *gros* topos is conceived as a category of spaces.

In fact, one advantage of *gros* toposes is that they are associated with sites which tend to have better categorical properties than those of the site presenting the *petit* topos.

Still, *gros* and *petit* toposes in a given pair are homotopically equivalent (as they are related by a local retraction), whence they share the same cohomological invariants.

The above result can be notably applied to construct pairs of *gros* and *petit* toposes starting from a (*K*-)full and (*K*-)faithful bimorphism of sites

$$(\mathcal{D}, K) \to (\mathcal{T}/T_{\mathcal{D}}, E_{T_{\mathcal{D}}}),$$

where $\mathcal{T}$ is a category endowed with a Grothendieck topology E, $T_{\mathcal{D}}$ is an object of $\mathcal{T}$ and $E_{T_{\mathcal{D}}}$ is the Grothendieck topology induced on $(\mathcal{T}/T_{\mathcal{D}})$ by E.

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Every Grothendieck topos is a 'small topos'

We define a Grothendieck topology $J^{\text{ét}}$ on **Topos**, which we call the étale cover topology, by postulating that a sieve on a topos $\mathcal{E}$ is $J^{\text{ét}}$-covering if and only if it contains a family $\{\mathcal{E}/A_i \to \mathcal{E} \mid i \in I\}$ of canonical local homeomorphisms such that the family of arrows $\{!_{A_i} : A_i \to 1_{\mathcal{E}} \mid i \in I\}$ is epimorphic in $\mathcal{E}$.

The functor L is a J-full and J-faithful bimorphism of sites

$$(\mathcal{C}, J) \to (\textbf{Topos}/\textbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\textbf{Sh}(\mathcal{C},J)}) \ .$$

So, by the above result, the 'petit' topos **Sh**$(\mathcal{C}, J)$ identifies with a coadjoint retract of the 'big' topos
Sh(**Topos**/**Sh**$(\mathcal{C}, J), J^{\text{ét}}_{\textbf{Sh}(\mathcal{C},J)}) \simeq$ **Sh**(**Topos**, $J^{\text{ét}})/I(\textbf{Sh}(\mathcal{C}, J))$ (in a suitable Grothendieck universe) via the local morphism **Sh**(L) and the essential inclusion C_L.

This shows that *every* Grothendieck topos can be naturally regarded as a 'petit' topos embedded in an associated 'gros' topos, and that this embedding allows one to view any object of the original topos as an étale morphism to the terminal object in the associated 'gros' topos, thus providing a solution to Grothendieck's problem.

Future developments

Our notion of relative site will play a key role in our future
development the theory of *relative toposes*.

We expect the development of this theory to parallel that of the
classical theory; indeed, by using a general stack semantics, we
plan to introduce, in a canonical, not *ad hoc* way, natural
generalizations to the relative setting of the classical notions of
morphism and comorphism of sites, flat functors, separating sets
for a topos, denseness conditions etc.

This will notably lead us to relative versions, in the language of
stacks (or, more generally, of indexed categories), of Giraud's and
Diaconescu's theorems, as well as to a theory of classifying
toposes of (higher-order) relative geometric theories.

Towards relative geometric logic

Indeed, the geometric approach to relative toposes which we have developed so far has a logical counterpart, which we may call relative geometric logic.

In its classical formulation, geometric logic does not have parameters embedded in its formalism; still, it is possible to introduce them without changing its degree of expressivity.

In a relative setting, parameters are fundamental if one wants to reason geometrically and use fibrational techniques. In fact, the semantics of stacks involves parameters in an essential way.

It turns out that the logical framework corresponding to relative toposes is a kind of fibrational, higher-order parametric logic in which it is possible to express a great number of higher-order constructions by using the parameters belonging to the base topos.

For further reading

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