Computability of Harmonic Measure

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Summary

- Introduction:
  - Dirichlet Problem and Harmonic Measure
  - Computability in $\mathbb{R}^n$
- Review of the main results
- Some ideas involved in the proofs

Dirichlet Problem and Harmonic Measure

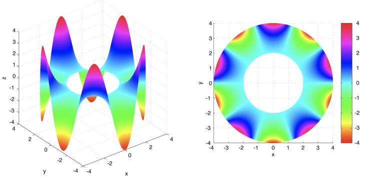
Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set (a domain).

The Dirichlet Problem

Given $f : \partial\Omega \longrightarrow \mathbb{R}$ continuous, find $u_f : \overline{\Omega} \longrightarrow \mathbb{R}$ s.t.

- u_f is continuous on $\overline{\Omega}$
- u_f is harmonic ($\Delta u_f = 0$) inside Ω
- $u_f = f$ on $\partial\Omega$.

Ex: conditions $f(r = 2) = 0$, $f(R = 4) = 4\sin(5\alpha)$:



Brownian Motion and Harmonic Measure

Harmonic Measure

Given $x \in \Omega$, let $\mathcal{B}_x^t$ be a Brownian motion started at x. The harmonic measure at x, ω_x^Ω, of a set $S \subset \partial\Omega$, is defined by:

$$\omega_x^\Omega(S) = \mathbb{P}[\mathcal{B}_x^t \text{ first exit } \Omega \text{ through } S].$$

Example: when Ω is simply connected, recall that for a given $x \in \Omega$, there is a unique conformal map $\phi_x : \mathbb{D} \to \Omega$ satisfying:

- $\phi_x(0) = x$
- $\phi_x'(x) > 0$

Then ω_x^Ω is the push-forward by (the extension of) ϕ_x of Lebesgue measure:

$$\omega_x^\Omega(S) = \text{Leb}(\phi_x^{-1}(S)).$$

Regular domains

A point $z \in \partial\Omega$ is called *regular* if $\forall n \in \mathbb{N},\ \exists \varepsilon > 0$ s.t. if $x \in \Omega$, then

$$|x - z| < \varepsilon \implies \omega_x^\Omega(B(x, 2^{-n})) > 1 - 2^{-n}$$

A domain is *regular* if all its points are regular (“no dust” is allowed).

Harmonic Measure as solutions to Dirichlet Problem

Let Ω be a **regular domain**. By a famous theorem of Kakutani, given a continuous $f : \partial\Omega \to \mathbb{R}$, the function u_f defined by:

$$u_f(x) = \int_{\partial\Omega} f(z)\, d\omega_x^\Omega$$

is harmonic in Ω and satisfies

$$\lim_{x \to z \in \partial\Omega} u_f(x) = f(z)$$

for all $z \in \partial\Omega$.

The extended function is therefore a solution to the Dirichlet Problem.

The questions we ask

For a regular domain Ω:

- what data about Ω do we need in order to compute ω_x^Ω ?
- what can we learn about Ω from the ability to sample the harmonic measure?

- For an open set $U \subset \mathbb{R}^n$, we say it is *lower computable (l.c.)* if there is a uniformly computable sequence of rational balls (B_n) s.t.

$$U = \bigcup_n B_n.$$

- For a closed set $E \subset \mathbb{R}^n$, we say it is:
  - *upper computable* if its complement is l.c.
  - *lower computable* if the collection

    $$\mathcal{B}_E = \{B(q,r) : B(q,r) \cap E \neq \emptyset\}$$

    is recursively enumerable.
  - *computable* if it is both upper and lower computable.

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function and $x_o \in D$. We say that:

- the value $f(x_o)$ is *computable relative to* x_o, if there is an algorithm which, upon input $\varepsilon \in \mathbb{Q}$, if provided with arbitrarily good approximations of x_o (by an oracle), halts and outputs an ε approximation of $f(x_o)$.

- the function f is *piece-wise computable*, if for each $x \in D$, the value $f(x)$ is computable relative to x (with possibly different algorithms in different pieces).

- the function f is *computable* if it is piece-wise computable with only one piece.
  ($f(x)$ is computable relative to x, *uniformly*).

a probability measure μ with support in $\mathbb{R}^n$ is *computable* if for any sequence

$$f_1, f_2, f_3, \ldots$$

of uniformly computable functions, the integrals

$$\int f_i \, d\mu$$

are uniformly computable.

- Ex: Lebesgue or any measure with a computable density.
- Remark: the support of a computable measure is a lower computable closed set.

All learn from one

What can we learn from the ability to sample harmonic measure at one single point x_0?

Theorem

Let Ω be a regular domain such that $\omega^{\Omega}_{x_0}$ is computable relative to x_0 for some computable $x_0 \in \Omega$. Then:

- *ω^{Ω}_{x} is computable relative to x, for all $x \in \Omega$,*
- *however, the computability of $(\omega^{\Omega}_{x})_x$ is not necessarily uniform,*
- *if Ω is lower computable, then $(\omega^{\Omega}_{x})_x$ is uniformly computable.*

open question: is *every* domain with uniformly computable harmonic measure, lower computable?

Corollary

There exist a regular domain Ω *and a computable boundary condition* $f : \partial\Omega \to \mathbb{R}$ *s.t.*

- *the unique solution to Dirichlet problem* $u_f(x)$ *is piece-wise computable,*
- *but* u_f *is not a computable function.*

What data about Ω do we need?

pure topological data is not enough.

Theorem

There exists a regular domain Ω such that:

- Ω *is lower computable*
- $\partial\Omega$ *is computable*
- *the harmonic measure* ω_x^Ω *is not computable for any* $x \in \Omega$

... some additional property/information is required.

A sufficient geometric/probabilistic condition

Definition

A domain Ω is called *computably regular* if there is a computable function $\varepsilon(n)$ s.t. for all $z \in \partial\Omega$ and $x \in \Omega$:

$$|z - x| < \varepsilon(n) \implies \omega_x^\Omega(B(x, 2^{-n})) > 1 - 2^{-n}$$

Theorem

Let Ω be a regular domain with a computable boundary $\partial\Omega$. Then, TFAE:

- *Ω is computably regular*
- *ω_x^Ω is computable for some $x \in \Omega$*
- *ω_x^Ω is uniformly computable for all x.*

FIN